MODERN JUNIOR MATHEMATICS

BOOK THREE

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PREFACE

The old order in mathematics teaching is rapidly giving way to a newer one more interesting, more vital, and more effective. Formerly, all phases of arithmetic were taught in the seventh and eighth grades. In the ninth grade, the foundations of algebra were laid. The latter had practically no connection with the arithmetic that came before nor with the geometry that came after. It was mostly the juggling of symbols that symbolized nothing. This algebra took on some meaning later for the few who continued the study of mathematics in higher schools. But for the many, it never functioned.

With the organization of the junior high schools has come a reorganization of mathematics. It is now taught in cycles, each complete in itself and adapted to the needs and abilities of the pupil, regardless of whether he continues the study of mathematics in school or applies it in the office, store, or shop. The purpose of the junior cycle is to give the pupil a broad knowledge and usable power and skill in the field of elementary mathematics. This cannot be done by the old tandem courses of arithmetic, algebra, geometry, and trigonometry. Nor will alternate bits of formal algebra, geometry, and trigonometry solve the problem. The result is a mastery of none and a confusion of all.

In this series the elements of arithmetic, geometry, algebra, and trigonometry are taught as one subject. Book One is largely arithmetical, but it uses the graph and the formula. Book Two is largely geometric, but it extends the pupil's knowledge of arithmetic through its practical appli-

cations in mensuration and it introduces general number in a way that makes algebraic symbols really symbolize. Book Three is largely algebraic, but new meaning is given to arithmetic number relations and processes, while geometry continues to give meaning to algebra and offers with trigonometry an interesting field of application.

The definite aims of this study are:

1. To extend the pupil's knowledge of the fundamental processes in algebra to negative expressions.
2. To develop an appreciation of modern algebraic language by giving the symbols a real meaning and by showing how these symbols grew.
3. To make the equation and its manipulation so familiar that the pupil will use it naturally as a convenient tool.
4. To enable him to make and interpret formulas.
5. To give him an inkling of the functional idea.
6. To continue to illuminate all mathematics through graphic representation.
7. To give the pupil a glimpse into one of the most interesting fields of applied algebra, namely trigonometry.
8. To let him discover the labor-saving device in logarithms.
9. To give a brief introduction to the logic of demonstrative geometry.
10. To give him a knowledge of elementary mathematics so wide that he may know something of how mathematics functions in the work of the world; a knowledge so definite that he has masterly skill in the use of a few of its tools; and a knowledge so meaningful that it may function for him whenever a later need arises, whether in the drafting room, the shop, the factory, or the senior high school.

The book is planned for a year's work in the ninth grade. the last year of the junior high school or the first of the high school on the 8–4 plan. It presupposes a knowledge of the introduction to algebra given through the mensuration of Book Two. With a little supplementation, however, it may be taught independently of Book Two.

The author desires to express her sincere appreciation to Dr. David Eugene Smith of Columbia University for his kindness in extending to her the privileges of his library of rare mathematics books. She is very much indebted to Miss Amy F. Preston of Roosevelt School, Columbus, for her continued assistance and experimentation. She hereby acknowledges her indebtedness also to Miss Alva Edwards of Mound Street Intermediate School and to Mr. J. E. Newell of Avondale Intermediate School, Columbus, for reading the manuscript and for giving helpful suggestions. Sincere appreciation is extended to many co-workers in Columbus for their encouragement and inspiration.

MARIE GUGLE

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MODERN JUNIOR MATHEMATICS

BOOK THREE

CHAPTER ONE

A NEW KIND OF NUMBER

A. THE MEANING OF OPPOSITES

In all your previous study of mathematics, you have used numbers beginning with zero (0), which are counted up the scale, as follows: 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, . . . With these numbers you have measured distances up and down, distances north and south, time going and time returning, time before and after the birth of Christ, money gained and money lost, money saved and money spent, and the degrees the temperature rises and the degrees it falls.

To show these measures of opposites you have used words to indicate which direction or which kind of measure was meant. Mathematicians have found it more convenient to use signs with the numbers to show these opposites. Before finding out what these signs are, let us be sure we know just what is meant by opposites.

1. What are the opposites of the following ideas?

	Idea	Opposite		Idea	Opposite
(a)	up	. . . . .	(i)	above sea level	. . . . .
(b)	east	. . . . .	(j)	bank deposit	. . . . .
(c)	gain	. . . . .	(k)	time A.D.	. . . . .
(d)	out	. . . . .	(l)	north latitude	. . . . .
(e)	height	. . . . .	(m)	forward	. . . . .
(f)	above zero	. . . . .	(n)	addition	. . . . .
(g)	increase	. . . . .	(o)	plus	. . . . .
(h)	right	. . . . .	(p)	positive	. . . . .

2. If all of the foregoing ideas are considered *positive* ideas, what one name might you give to all their opposites?

3. If the sign *plus* belongs with the positive ideas, what sign would naturally show their opposites, or negative ideas?

B. POSITIVE AND NEGATIVE NUMBERS

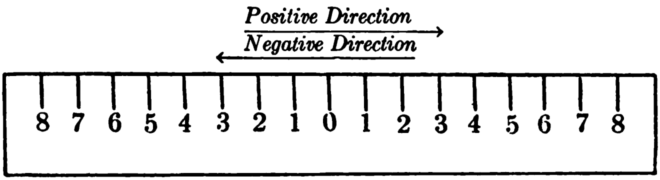
I. Their Meaning

1. If a thermometer registers 38 degrees above zero, we say the temperature is 38°. Since this is a temperature above zero, it is positive 38°, or + 38°. If the temperature falls 38°, the mercury stands at 0°. If the temperature falls 8° more, the mercury stands at 8° below zero, that is, at negative 8° or − 8°.

If the mercury rises, it moves in the positive direction; if it falls, it moves in the negative direction.

From the thermometer we see that the scale of numbers may extend in both directions from zero.

2. Instead of writing the number scale vertically, as on the thermometer, we may write it horizontally, as in the figure below.



HORIZONTAL SCALE OF NUMBERS

3. On the horizontal scale, numbers to the *right* of zero are *positive* and those to the *left* are *negative*.

4. The number -2 may be read "*negative two*" or "*minus two.*"

5. If a number has no sign, the positive number is meant. 2 or $+2$ may be read "*positive two*" or "*plus two.*"

6. On the scale, locate the following numbers: $+3$, -3, $+1$, -5, -8, 10, -1, 6, -7, -9.

II. Addition of Positive and Negative Numbers

1. (a) If the mercury is at 10° above zero and the temperature rises 15°, where will the mercury be? Where will it be if the temperature falls 15°?
   (b) On a horizontal scale begin at positive 10 and count 15 in the positive direction. What number is reached?
   (c) Count 15 in the negative direction and name the point reached.

2. (a) If the mercury is at 10° below zero and the temperature rises 15°, where will the mercury be? Where will it be if the temperature falls 15°?
   (b) On the horizontal scale, begin at negative 10 and count 15 in the positive direction. What number is reached?
   (c) Count 15 in the negative direction and name the point reached.

3. (a) Counting upward on the vertical scale, or to the right on the horizontal, is adding a positive number.
   (b) In what direction must you count to add a negative number?

4. Write these statements in equation form:
   (a) Positive 10 plus positive 15 equals positive 25.
   $$(+10) \quad + \quad (+15) \quad = \quad +25$$
   (b) Positive 10 plus negative 15 equals negative 5.
   $$(+10) \quad + \quad (-15) \quad = \quad -5$$

(c) Negative 10 plus positive 15 equals positive 5.

$$(-10) \quad + \quad (+15) \quad = \quad +5$$

(d) Negative 10 plus negative 15 equals negative 25.

$$(-10) \quad + \quad (-15) \quad = \quad -25$$

5. These additions may be put in the usual arithmetic form thus:

$$\begin{array}{r} +10 \\ +15 \\ \hline +25 \end{array} \qquad \begin{array}{r} +10 \\ -15 \\ \hline -\ 5 \end{array} \qquad \begin{array}{r} -10 \\ +15 \\ \hline +\ 5 \end{array} \qquad \begin{array}{r} -10 \\ -15 \\ \hline -25 \end{array}$$

6. (a) When both numbers are positive, what must you do with them to find the sum? What sign does the sum have?

(b) When both numbers are negative, how do you find the sum?

(c) When one number is positive and one is negative, what must you do with them to find the sum? Does it make any difference if the upper number is the smaller? Which sign does the sum have?

III. Problems in Addition of Signed Numbers

Write the solutions of the following problems in three ways: In an English statement; as an equation; and in the form of arithmetic addition.

1. If you walk 5 blocks east and then 2 blocks west, how far are you from the starting point and in which direction from it?

2. Find your location if you walk 6 blocks north and 8 blocks south.

3. (a) Find the location of a ship that sails 7° north and 9° south.

(b) Find its location if it starts out from a place 1° north latitude and sails 7° north and 9° south.

(c) Find its location if it starts from a place 10° south latitude and sails south 10°, then north 15°.

4. How much are you worth if you have \$85 in the bank and \$10 cash in your pocket?

5. How much are you worth if you have \$15 and owe \$13 for clothes?

6. How much are you worth if you have \$15 and buy a \$20 suit on credit?

7. If you have \$65 in the bank and have \$15 cash, how much are you worth if you owe \$20 for a suit and \$8 for shoes?

8. On Monday a pupil's rating was 15 % above passing grade. On Tuesday his rating was 5 % lower than on Monday. On Wednesday it increased 10 %. On Thursday it decreased 30 %, and on Friday it increased 20 %. What was his grade on Friday if 75 % is passing grade? Draw a line graph of his week's work.

Suggestion: This rating may be found (a) by finding each day's rating throughout the week; or (b) by finding the sum of all the positive numbers and the sum of all the negative numbers, and then combining the two results.

$$(+15) + (-5) + (+10) + (-30) + (+20) = +10$$

$$\text{(a)}\quad \begin{array}{r} +15 \\ -\ 5 \\ +10 \\ -30 \\ \underline{+20} \\ +10 \end{array} \qquad\qquad \text{(b)}\quad \begin{array}{r} +15 \\ +10 \\ \underline{+20} \\ +45 \end{array} \qquad \begin{array}{r} -\ 5 \\ \underline{-30} \\ -35 \end{array} \qquad \begin{array}{r} +45 \\ \underline{-35} \\ +10 \end{array}$$

$75\,\% + 10\,\% = 85\,\%$, Friday's grade.

9. Make up a problem similar to each of the first eight and solve your own or those of a classmate.

10. Make up five problems showing different uses of positive and negative numbers.

11. A football game starts with the ball at the center of the field. During the play, it is carried sometimes toward one goal and sometimes toward the other. Make a diagram of three successive plays and give the location of the ball.

12. Find the following sums:

(a)	(b)	(c)	(d)
+ 12	− 12	+ 12	− 12
+ 17	− 17	− 17	+ 17

(e)	(f)	(g)	(h)
− 4	+ 6	− 18	− 25
− 9	− 5	− 23	+ 19
+ 15	− 9	+ 6	− 40
− 2	+ 3	+ 30	+ 7

13. Find the following sums:

Write results only. Keep a record of the time required. Practice on these or similar exercises until you can write the sums speedily and accurately.

(a)	(b)	(c)	(d)
+ 6	+ 7	− 16	− 21
+ 3	− 9	+ 32	− 32
− 7	+ 15	+ 14	− 18
− 8	− 8	− 19	+ 25

(e)	(f)	(g)	(h)
+ 7	+ 7	− 35	+ 13
− 6	− 2	+ 42	+ 19
− 5	− 7	− 17	+ 31
+ 11	+ 2	+ 56	− 42

(i)	(j)	(k)	(l)
− 5	− 6	+ 4	− 6
+ 4	− 8	− 6	− 9
+ 6	+ 9	+ 8	+ 3
− 8	+ 7	− 5	− 6
− 9	+ 3	− 9	+ 8

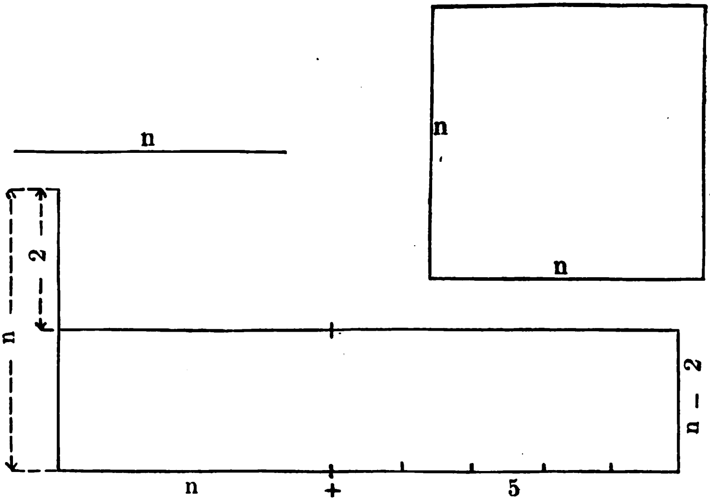
(m)	(n)	(o)	(p)
+ 27	+ 31	− 23	+ 62
− 13	− 26	+ 47	+ 13
+ 42	+ 12	+ 12	− 72
− 18	− 45	− 21	+ 23
+ 21	− 51	− 27	− 35

14. (a) Draw an unmeasured line. Draw a square on the line. If its length is n cm. find its perimeter.

(b) Draw a rectangle 5 cm. longer than the square and 2 cm. narrower than the square.

What is its length?. Its width? What is its perimeter?

(c) *Solution:*



$$P_{\square} = 2(l + w) = 2(n + 5 + n - 2) = 2(2n + 3) = 4n + 6$$

or

$$P_{\square} = 2l + 2w = 2(n + 5) + 2(n - 2) = 2n + 10 + 2n - 4 = 4n + 6$$

(d) Measure the line n and find the number of centimeters in the perimeter of the rectangle.

(e) *Explanation:*

The perimeter might be found by adding the lengths of the four sides, thus:

$$\begin{array}{r} \left.\begin{array}{r} n+5 \\ n+5 \end{array}\right\} \text{sides} \\ \left.\begin{array}{r} n-2 \\ n-2 \end{array}\right\} \text{ends} \\ \hline 4n+6 \end{array}$$

This form shows that the sum of the two ends of the rectangle is $2n - 4$. We see, therefore, that $2(n - 2) = 2n - 4$. Or -2 multiplied by $+2$ gives -4.

(f) *Other illustrations:*

(1) If you lost \$2 on Monday and \$2 on Tuesday, how much did you lose on both days?

2 times -2 dollars equal -4 dollars.

$2(-2) = -4$.

(2) If the temperature dropped 3 degrees an hour for four hours, how many degrees did it drop?

4 times -3 degrees equal -12 degrees.

$4(-3) = -12$.

15. (a) Draw another unmeasured line x in. long.

(b) Draw a rectangle $x + 2$ in. long and $x - 3$ in. wide.

(c) Find the perimeter in three ways.

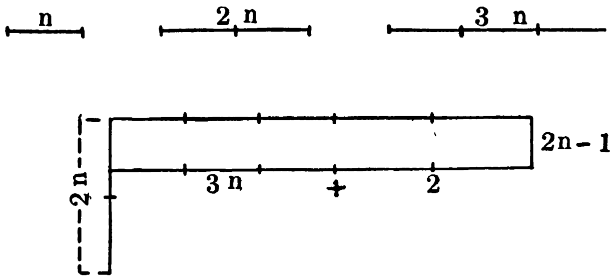
(d) Measure x and find the number of inches in the perimeter.

16. (a) From the following dimensions find the perimeter of each rectangle.

(b) Give a reasonable measure to each line and find the numerical length of each perimeter.

No.	Length	Width	Perimeter	Numerical Length: Of Line	Numerical Length: Of Perimeter
(1)	$n+4$	$n-5$	$4n-2$	If $n = 8$ ft.	30 ft.
(2)	$a+12$	$a-9$	......	....	....
(3)	$x+16$	$x-10$	......	....	....
(4)	$y+3\frac{1}{2}$	$y-2$	......	....	....
(5)	$b+1\frac{5}{8}$	$b-2\frac{3}{8}$	......	....	....
(6)	$m+6.6$	$m-8.2$	......	....	....
(7)	$r+2\frac{1}{2}$	$r-3\frac{3}{4}$	......	....	....
(8)	$z+3.5$	$z-4.75$	......	....	....

17. (a) Draw any unmeasured line, n in. long.
Draw a square on $2n$; on $3n$.
Draw a rectangle $3n + 2$ in. long and $2n - 1$ in. wide.



(Scale ½)

(b) Find the perimeter of the rectangle in three ways.
(1) By formulas.

$$
\begin{aligned}
P_{\square} &= 2(l+w) \\
&= 2(3n+2+2n-1) \\
&= 2(5n+1) \\
&= 10n+2
\end{aligned}
\qquad \text{or} \qquad
\begin{aligned}
P_{\square} &= 2l+2w \\
&= 2(3n+2)+2(2n-1) \\
&= 6n+4+4n-2 \\
&= 10n+2
\end{aligned}
$$

(2) By addition.

$$\begin{array}{r} 3n + 2 \\ 3n + 2 \\ 2n - 1 \\ \underline{2n - 1} \\ 10n + 2 \end{array}$$

(c) Find the numerical value of the perimeter. By measuring, n is found to be $\frac{3}{4}$ of an inch. Substitute $\frac{3}{4}$ of an inch for n in the equation of the perimeter.

$$\begin{aligned} P_{\square} &= 10n + 2 \\ &= 10(\tfrac{3}{4}) + 2 \\ &= \tfrac{15}{2} + 2 \\ &= 7\tfrac{1}{2} + 2 \\ &= 9\tfrac{1}{2} \text{ in.} \end{aligned}$$

18. (a) From the dimensions given in the table find the perimeter of each rectangle.

No.	Length	Width	Perimeter	Numerical Length	
				Of Line	Of Perimeter
(1)	$2n + 5$	$2n - 3$	......	If $n = 2$ in.	....
(2)	$5n + 1$	$4n - 1$	......	If $n = 3$ ft.	....
(3)	$3x + 7$	$2x - 6$	......	If $x = 4$ cm.	....
(4)	$3a - 4$	$a + 1$	......	If $a = 5$ yd.	....
(5)	$2r + 3$	$2r - 1$	......	If $r = 3$ rd.	....
(6)	$7y + 1$	$7y - 1$	......	If $y = 2$ ft.	....
(7)	$5z + 3$	$5z - 3$	......	If $z = 1$ in.	....
(8)	$10b + 7$	$8b - 5$	......	If $b = 1\frac{1}{2}$ in.	....
(9)	$5x + 4$	$5x - 3$	......	If $x = 1.2$ in.	....
(10)	$2x + 7$	$x - 3$	......	If $x = 5$ ft.	....
(11)	$5w - 2$	$3w - 1$	......	If $w = 2$ yd.	....
(12)	$6z - 1$	$5z - 2$	......	If $z = 1$ in.	....
(13)	$15c - 2$	$15c - 4$	......	If $c = 1$ in.	....
(14)	$20a - 3$	$16a - 3$	......	If $a = \frac{3}{4}$ in.	....
(15)	$13d - 7$	$11d - 5$	......	If $d = 2$ ft.	....
(16)	$24x - 3$	$24x - 5$	......	If $x = 1$ ft.	....
(17)	$12m + 7$	$8m - 3$	......	If $m = 1\frac{3}{4}$ in.	....
(18)	$14y - 3$	$12y - 1$	......	If $y = 1\frac{1}{2}$ in.	....
(19)	$27x - 12$	$21x - 7$	......	If $x = 2$ ft.	....
(20)	$36r - 5$	$32r - 3$	......	If $r = 1\frac{1}{4}$ ft.	....

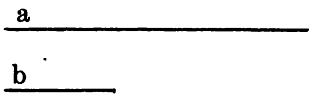
(b) In the first problem, use the first formula for perimeter. In the second problem, use the second formula. In the third problem, find the perimeter by adding the lengths of the four sides. Use the three different methods in succession for the other problems.

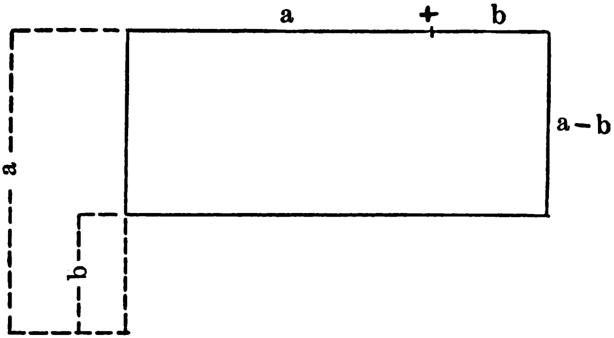
(c) Find the numerical length of each perimeter by using the given length for each line.

19. (a) Draw two unmeasured lines. Let the longer one be a inches long and the shorter one b inches long.

(b) Draw a line equal to $a + b$; $a - b$; $2a + b$; $2a - b$; $3a - 2b$; $5a - 3b$.

(c) Draw a rectangle $a + b$ inches long and $a - b$ inches wide.





(d) Find the perimeter of the rectangle.

$$\begin{aligned} P_{\square} &= 2(l + w) \\ &= 2(a + b + a - b) \\ &= 2(2a) \\ &= 4a \end{aligned}$$

or

$$\begin{aligned} P_{\square} &= 2l + 2w \\ &= 2(a + b) + 2(a - b) \\ &= 2a + 2b + 2a - 2b \\ &= 4a \end{aligned}$$

(e) *Suggestion:* Just as $+2$ and -2 offset each other, so $+b$ and $-b$ or $+2b$ and $-2b$ offset each other.

Positive and negative numbers with letters are added in the same way that arithmetical numbers are.

$$\begin{array}{r} +5 \\ +8 \\ \hline +13 \end{array} \qquad \begin{array}{r} +5 \\ -8 \\ \hline -3 \end{array} \qquad \begin{array}{r} +5\text{ ft.} \\ -8\text{ ft.} \\ \hline -3\text{ ft.} \end{array} \qquad \begin{array}{r} +5a \\ -8a \\ \hline -3a \end{array} \qquad \begin{array}{r} -5a^2 \\ +8a^2 \\ \hline +3a^2 \end{array} \qquad \begin{array}{r} -5n \\ -8n \\ \hline -13n \end{array}$$

(f) Find the perimeter by adding the 4 sides.

$$\begin{array}{r} a+b \\ a+b \\ a-b \\ a-b \\ \hline 4a \end{array}$$

On a straight line lay off with a compass the lengths of the four sides. On the same line lay off $4a$ and compare.

(g) Draw a rectangle $5a - 3b$ inches long and $3a - 2b$ inches wide. Find the perimeter.

(1) By formula.

$$\begin{aligned} P_{\square} &= 2(l + w) \\ &= 2(5a - 3b + 3a - 2b) \\ &= 2(8a - 5b) \\ &= 16a - 10b \end{aligned} \qquad \text{or} \qquad \begin{aligned} P_{\square} &= 2l + 2w \\ &= 2(5a - 3b) + 2(3a - 2b) \\ &= 10a - 6b + 6a - 4b \\ &= 16a - 10b \end{aligned}$$

(2) By addition.

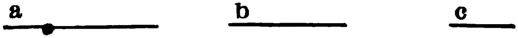
$$\begin{array}{r} 5a - 3b \\ 5a - 3b \\ 3a - 2b \\ 3a - 2b \\ \hline 16a - 10b \end{array}$$

(h) Measure lines a and b and find the numerical length of the perimeter.

By measurement, $a = 4$ cm. and $b = 1\frac{1}{2}$ cm.

$$\begin{aligned} P_{\square} &= 16a - 10b \\ &= 16(4) - 10(1\tfrac{1}{2}) \\ &= 64 - 15 \\ &= 49 \text{ cm.} \end{aligned}$$

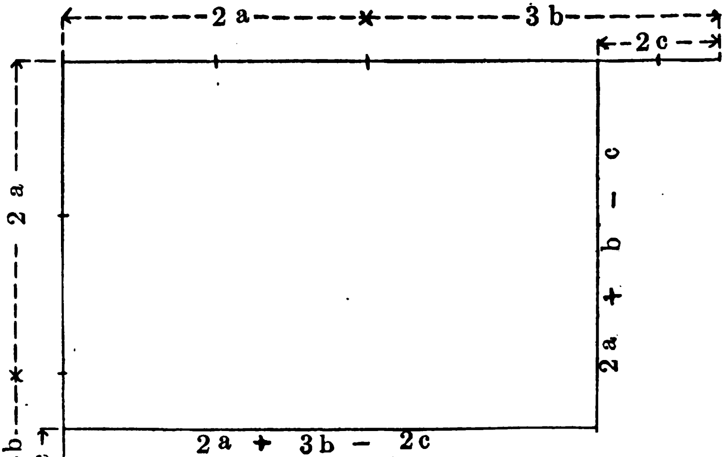
20. (a) Draw three lines of different lengths, a, b, and c.



(Scale $\frac{1}{2}$)

(b) Draw a line equal to $a + b + c$; $a + b - c$; $2a + b - c$; $2a + 3b - 2c$; $3a - 2b + c$; $4a - b - c$.

(c) Draw a rectangle whose length is $2a + 3b - 2c$ and whose width is $2a + b - c$.



(Scale $\frac{1}{2}$)

(d) Find the perimeter of the rectangle.

(1) By formula.

$$\begin{aligned} P_{\square} &= 2(l + w) \\ &= 2(2a + 3b - 2c + 2a + b - c) \\ &= 2(4a + 4b - 3c) \\ &= 8a + 8b - 6c \end{aligned}$$

Or, $P_{\square} = 2l + 2w$

$$= 2(2a + 3b - 2c) + 2(2a + b - c)$$
$$= 4a + 6b - 4c + 4a + 2b - 2c$$
$$= 8a + 8b - 6c$$

(2) By adding the lengths of the four sides.

$$\begin{array}{rl} & 2a + 3b - 2c \\ & 2a + 3b - 2c \\ & 2a + b - c \\ & \underline{2a + b - c} \\ P_{\square} = & 8a + 8b - 6c \end{array}$$

(e) Measure a, b, and c, and find the numerical length of the perimeter. By measurement, $a = 4$ cm., $b = 3$ cm., and $c = 1\frac{1}{2}$ cm.

$$P_{\square} = 8a + 8b - 6c$$
$$= 8(4) + 8(3) - 6(1\tfrac{1}{2})$$
$$= 32 + 24 - 9$$
$$= 47 \text{ cm.}$$

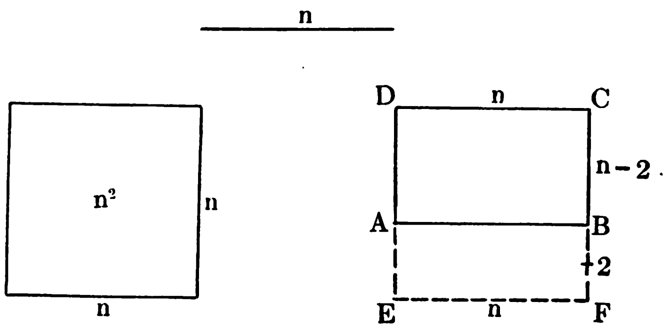
21. (a) From the dimensions given in the table find the perimeters of each rectangle.

No.	Length	Width	Perimeter	Numerical Lengths: Of Lines	Numerical Lengths: Of Perimeter
(1)	$2a + 3b$	$2a - b$	....	$a = 2,\quad b = 1$	....
(2)	$3x + 2y$	$2x - y$	....	$x = 3,\quad y = 2$	....
(3)	$4m + 3n$	$2m - 2n$	....	$m = 3,\quad n = 1$	....
(4)	$5k + l$	$k - l$	....	$k = 2.2\quad l = 1$	....
(5)	$3x + 7y$	$2x - 3y$	....	$x = 5,\quad y = 2$	....
(6)	$6a + 4b$	$3a - 2b$	....	$a = 4,\quad b = 3$	....
(7)	$8x + 3y$	$4x - y$	....	$x = 1\frac{1}{2},\quad y = 2$	....
(8)	$4a - b - c$	$3a - 2b + c$	....	$a = 3,\quad b = 2,\quad c = 1$	....
(9)	$5x - y - z$	$2x - y + z$	....	$x = 2,\quad y = 1,\quad z = 1$	....
(10)	$3d - 2e - f$	$2d - e - f$	....	$d = 4,\quad e = 3,\quad f = 2$	....
(11)	$a + 2b - c$	$3a + c$	....	$a = 2,\quad b = 5,\quad c = 1$	....
(12)	$x + 3y - z$	$2x + z$	....	$x = 3,\quad y = 2,\quad z = 1$	....
(13)	$2r + s - 2t$	$3s + 2t$	....	$r = 5\frac{1}{2},\quad s = 2,\quad t = 1\frac{1}{2}$	....
(14)	$3h + 6t - 5u$	$4h + 2u$	....	$h = 4,\quad t = 3,\quad u = 2$	....
(15)	$a + 3b - 2c$	$a + b$	....	$a = 4,\quad b = 2,\quad c = 1$	....
(16)	$4b + 2c - d$	$2b - c$	....	$b = 2,\quad c = 3,\quad d = 1$	....
(17)	$5x + 2y - 3z$	$4x + y$	....	$x = 5,\quad y = 4,\quad z = 2$	....
(18)	$10m + 3n - k$	$4m + k$	....	$m = \frac{1}{2},\quad n = \frac{1}{3},\quad k = 1$	....
(19)	$6a + 2b - 3c$	$2a - 2b$	....	$a = 4,\quad b = 2,\quad c = 2$	....
(20)	$4x - y + 5z$	$3x + 2y$	....	$x = 1,\quad y = 1,\quad z = 1$	....

(b) Use the three different ways in succession.
(c) Find the numerical length of each perimeter, using the given lengths for the lines.

IV. Multiplication of Signed Numbers

1. (a) Draw a square n centimeters long.



(Scale ½)

(b) Draw another square $EFCD$ on the line n. What is its area?
(c) On the side DE, find point A, 2 cm. from E. On CF find point B, 2 cm. from F. Draw AB.
(d) What kind of figure is $ABCD$?
What kind is $EFBA$?
What is the area of $EFBA$?
(e) What figure is left when ▭ $EFBA$ is taken away from the square?

The ▭$ABCD$ = □ n^2 less the ▭ $2n$
$= n^2 - 2n$

But the ▭$ABCD$ $= lw$
$= n(n - 2)$
$= n^2 - 2n$

$$\begin{array}{l} n \quad - 2 \\ \underline{n \qquad\quad} \\ n^2 - 2n \end{array}$$

Note: $n(-2)$ is $-2n$, just as $2(-2)$ is -4.

(f) By measurement, n is found to be 5 cm. Find the area of $ABCD$ in square centimeters.

	Check
$S_{\square} = n^2 - 2n$	$n = 5$
$= 25 - 10$	$n - 2 = 3$
	$n(n - 2) = 5 \cdot 3$
$= 15$ sq. cm.	$= 15$

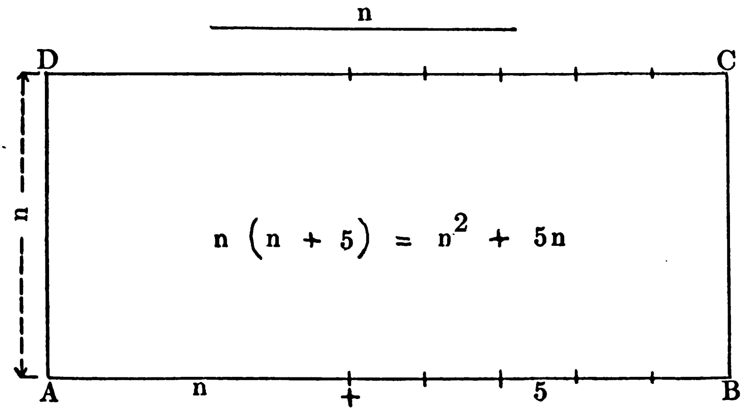
2. (a) Draw ten lines of different lengths and draw a square on each.

(b) In each measure off a part from one dimension and draw a rectangle narrower than the square.

(c) Find the area of each rectangle thus formed.

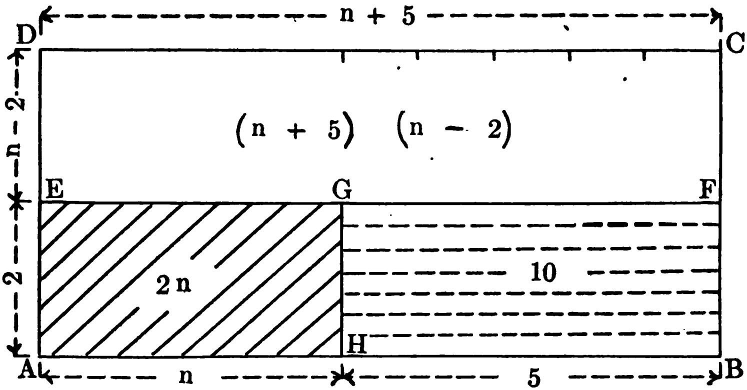
(d) Measure the side of the square and find the numerical measure of the area.

3. (a) Draw a rectangle whose dimensions are $n + 5$ and n centimeters.



Evidently the area of this rectangle $ABCD$ is $n^2 + 5n$.

(b) Cut off 2 cm. from the width and draw a line EF parallel to AB.



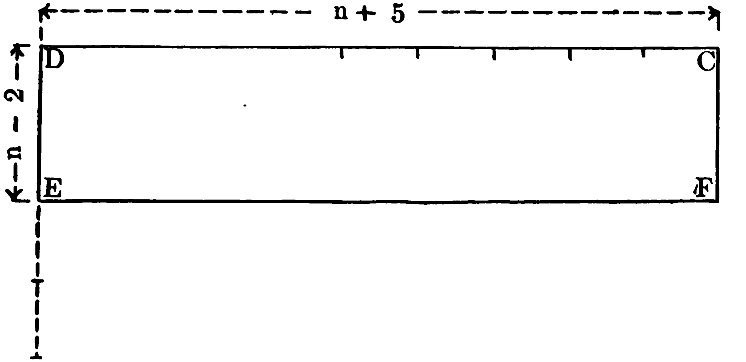
(c) At point H, 5 cm. from B, draw $GH \parallel BF$. What kind of figures are $AHGE$ and $HBFG$?

(d) If both of these rectangles are taken away from $\square ABCD$ what figure is left? What are the dimensions of $\square EFCD$? If from the area of the whole figure, $n^2 + 5n$, we take away the area of the two small rectangles, $2n$ and 10, we shall have left the area of $\square EFCD$.

That is, $\square EFCD = n^2 + 5n - 2n - 10$

$$= n^2 + 3n - 10$$

(e) The area of this rectangle may be found more easily by multiplying its length by its width.



(1) *Solution:*	*Process*
$S_\square = lw$	$n + 5$
$= (n + 5)(n - 2)$	$n - 2$
$= n^2 + 3n - 10$	$n^2 + 5n$
	$ - 2n - 10$
	$n^2 + 3n - 10$

(2) *Explanation:*

The $n + 5$ is multiplied first by n, which gives the partial product, $n^2 + 5n$. Then $n + 5$ is multiplied by -2, which gives $-2n - 10$. For just as n times -2 gives $-2n$, so -2 times n gives $-2n$. The product is the same no matter which factor is the multiplier.

It makes no difference whether we say 4 times 3 or 3 times 4, the product is the same either way. So it is when one number is negative; 5 times -2 or -2 times 5 both give a product of -10.

(f) Measure the length of n and find the area of the $\square EFCD$ in square centimeters.

By measurement, $n = 4$ cm.

	Check
$\therefore S_\square = n^2 + 3n - 10$	$l = n + 5 = 9$
$= (4)^2 + 3 \cdot 4 - 10$	$w = n - 2 = 2$
$= 16 + 12 - 10$	
$= 18$ sq. cm.	$lw = (n + 5)(n - 2) = 18$ sq. cm.

4. (a) Draw rectangles having the dimensions given below.

(b) Find the perimeter and area of each.

(c) Find the numerical value of the perimeter and area of each, using the given lengths for the lines.

No.	LENGTH	WIDTH	PERIMETER	AREA	NUMERICAL VALUE OF		
					Line	Perimeter	Area
(1)	$n + 6$	$n - 5$	....	....	$n = 7$	....	....
(2)	$x + 7$	$x - 3$	....	....	$x = 5$	....	....
(3)	$a + 5$	$a - 2$	....	....	$a = 6$	....	....
(4)	$b + 8$	$b - 6$	....	....	$b = 10$	....	....
(5)	$c + 10$	$c - 7$	....	....	$c = 9$	....	....
(6)	$2n + 5$	$n - 5$	....	....	$n = 5$	....	....
(7)	$3x + 7$	$x - 2$	....	....	$x = 4$	....	....
(8)	$4a + 6$	$a - 4$	....	....	$a = 6$	....	....
(9)	$3n + 2$	$2n - 3$	....	....	$n = 3$	....	....
(10)	$5a + 4$	$2a - 2$	....	....	$a = 5$	....	....
(11)	$7b + 3$	$3b - 4$	....	....	$b = 3$	....	....
(12)	$4c + 8$	$3c - 5$	....	....	$c = 6$	....	....
(13)	$8n + 5$	$6n - 4$	....	....	$n = 2$	....	....
(14)	$10x + 7$	$3x - 2$	....	....	$x = 4$	....	....
(15)	$5a + 6$	$2a - 3$	....	....	$a = 6$	....	....
(16)	$3m + 11$	$5m - 6$	....	....	$m = 4$	....	....
(17)	$3x + 6$	$4x - 8$	....	....	$x = 8$	....	....
(18)	$6n + 4$	$8n - 6$	....	....	$n = 1\frac{1}{2}$	....	....
(19)	$9s + 8$	$6s - 2$	....	....	$s = 1\frac{1}{3}$	....	....
(20)	$5x + 6$	$10x - 3$	....	....	$x = .8$	....	....

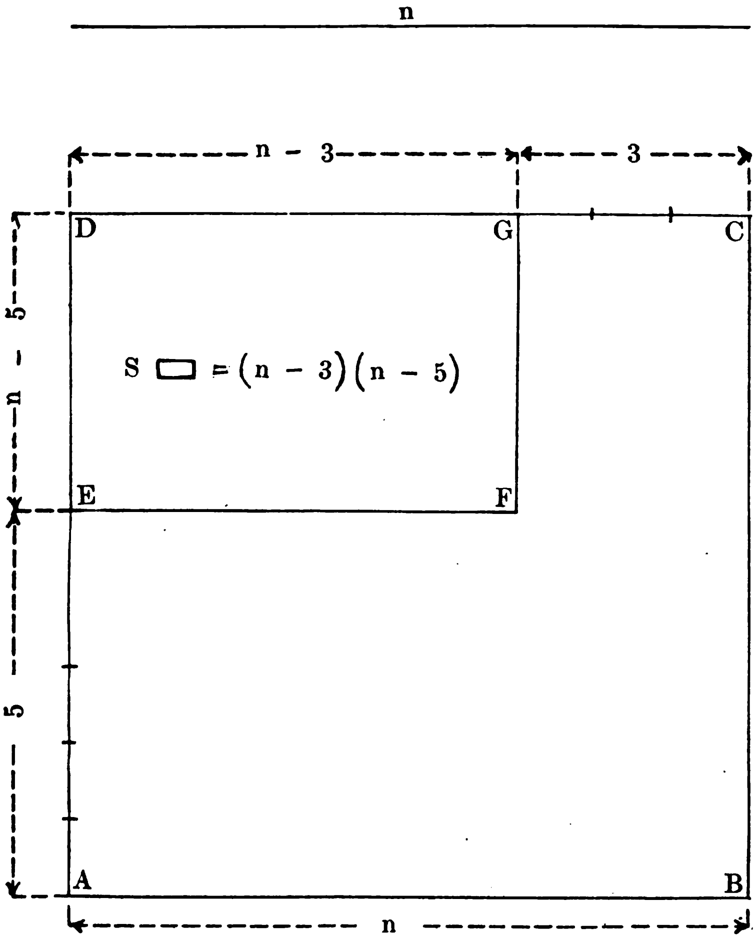
5. (a) Thus far, you have learned how to find the area of a rectangle, formed (1) by adding different numbers to the dimensions of a square, as $(n + 5)(n + 3)$; (2) by adding to the length and subtracting from the width, as $(n + 5)(n - 3)$.

(b) A rectangle may be formed by subtracting different numbers from both dimensions of a square.

(c) Draw a line n centimeters long and draw a square on that line, as $\square$ *ABCD*. What is the area of $\square$ *ABCD*?

(d) On *AD* find point *E*, 5 cm. from *A*; and on *CD* find *G*, 3 cm. from *C*. Draw the rectangle *EFGD*. How long is this rectangle? How wide is it?

(e) Its area, of course, is the product of the two dimensions, that is $(n - 3)(n - 5)$. To find this product, one must know how to multiply -3 by -5.



You have already learned to multiply numbers with signs in three ways. These are

(1) *plus* times *plus*, as + 3 times + 5.

$$\begin{array}{r} +\ 3 \\ \underline{+\ 5} \\ +15 \end{array}$$

(2) *plus* times *minus*, as $+3$ times -5.

$$\begin{array}{r} +\ 3 \\ -\ 5 \\ \hline -15 \end{array}$$

(3) *minus* times *plus*, as -3 times $+5$.

$$\begin{array}{r} -\ 3 \\ +\ 5 \\ \hline -15 \end{array}$$

The only other way is

(4) *minus* times *minus*, as -3 times -5.

To see what this product is, let us think of the meaning of numbers with signs.

(f) *Illustrations:*

(1) If five workingmen $(+5)$, able to earn 3 dollars a day $(+3)$, come into a town, the town is $+5$ times $+3$ dollars or $+15$ dollars better off.

(2) If these five men $(+5)$, *leave* the town with their earnings (-3), the town is 15 dollars worse off (-15).

(3) Five criminals, who not only earn nothing, but must be cared for in jail by the public, may be counted as negative. If five criminals come into a town (-5), and it costs the public 3 dollars a day to keep them in jail $(+3)$, the town is 15 dollars worse off (-15).

(4) But if these criminals (-5), were taken out of the town and the public were relieved of their expense (-3), then the town would be 15 dollars better off than before $(+15)$.

(5)
$$\begin{array}{r} +\ 3 \\ +\ 5 \\ \hline +15 \end{array} \qquad \begin{array}{r} -\ 3 \\ +\ 5 \\ \hline -15 \end{array} \qquad \begin{array}{r} +\ 3 \\ -\ 5 \\ \hline -15 \end{array} \qquad \begin{array}{r} -\ 3 \\ -\ 5 \\ \hline +15 \end{array}$$

(g) *Other illustrations:*

(1) The mercury in a thermometer is at 0°. It is rising 3° an hour (+ 3). Where will it stand 4 hours later (+ 4)?

$$\begin{array}{r} +\ 3 \\ \underline{+\ 4} \\ +\ 12 \end{array}$$, or 12° *above* zero.

(2) The mercury is at 0°. It is falling 3° an hour (– 3). Where will it stand 4 hours later (+ 4)?

$$\begin{array}{r} -\ 3 \\ \underline{+\ 4} \\ -\ 12 \end{array}$$, or 12° *below* zero.

(3) The mercury is at 0°. It has been rising 3° an hour (+ 3). Where was it 4 hours ago (– 4)?

$$\begin{array}{r} +\ 3 \\ \underline{-\ 4} \\ -\ 12 \end{array}$$, or 12° *below* zero.

(4) The mercury is at 0°. It has been falling 3° an hour (– 3). Where was it 4 hours ago (– 4)?

$$\begin{array}{r} -\ 3 \\ \underline{-\ 4} \\ +\ 12 \end{array}$$, or 12° *above* zero.

(h) (1) Examine these illustrations again.

(2) When the signs of both factors are *plus*, what is the sign of the product?

(3) When the signs of both factors are *minus*, what is the sign of the product?

(4) When the sign of one factor is *plus* and the sign of the other is *minus,* what is the sign of the product?

(i) Fill in the blanks correctly with either *plus* or *minus:*

(1) When the two factors have *like* signs, the sign of the product is ————.

(2) When the two factors have *unlike* signs, the sign of the product is ————.

(j) Read the following products until they can be given correctly without hesitation:

(1)	6	-6	-6	$+6$	-6	-7
	4	4	-4	-4	-8	5
(2)	-7	-5	-9	12	-15	13
	3	-8	6	-8	-5	-3
(3)	-16	-25	35	-18	-17	-24
	-3	5	-3	-5	2	-3
(4)	-18	15	-29	25	-32	-50
	$+6$	8	-4	-16	2	-25
(5)	n	$-2n$	$6n$	$7n$	$-10n$	$42n$
	-5	$-n$	n	$-2n$	$-15n$	$-10n$
(6)	$5t$	$-8t$	$-7t$	$+6t$	$+7t$	$-8t$
	$-2t$	$-2t$	$+8t$	$-9t$	$+7t$	$+8t$
(7)	$8a$	$-6b$	$-9n$	$18x$	$15k$	$+9y$
	$-6a$	$+4b$	$-5n$	$-6x$	$6k$	$-9y$
(8)	$-21c$	$+16n$	$-10k$	$31n$	$14h$	$-7a$
	$6c$	$-14n$	$-9k$	$-10n$	$-8h$	$+9a$

(k) (1) Find the area of the rectangle whose dimensions are $n - 3$ and $n - 5$.

$$\begin{aligned} S_{\square} &= lw \\ &= (n-3)(n-5) \\ &= n^2 - 8n + 15 \end{aligned} \qquad \begin{array}{r} n - 3 \\ n - 5 \\ \hline n^2 - 3n \qquad\quad \\ - 5n + 15 \\ \hline n^2 - 8n + 15 \end{array}$$

(2) Find the numerical value of the area when n is 9 cm.

$$\begin{aligned} S_{\square} &= n^2 - 8n + 15 \\ &= (9)^2 - 8 \cdot 9 + 15 \\ &= 81 - 72 + 15 \\ &= 24 \text{ sq. cm.} \end{aligned}$$

Check

$$l = n - 3 = 6$$
$$w = n - 5 = 4$$
$$lw = (n - 3)(n - 5) = 24 \text{ sq. cm.}$$

6. (a) From the following dimensions, find the perimeter and area of each rectangle.

No.	Length	Width	Perimeter	Area	Numerical Value of		
					Line	Perimeter	Area
(1)	$n - 4$	$n - 6$	.....	....	$n = 8$	.....	.....
(2)	$a - 5$	$a - 8$	.....	....	$a = 10$	.....	.....
(3)	$x - 9$	$x - 12$	.....	....	$x = 18$	.....	.....
(4)	$c - 7$	$c - 10$	.....	....	$c = 12$	.....	.....
(5)	$n - 2$	$n - 7$	.....	....	$n = 9$	.....	.....
(6)	$x - 8$	$x - 12$	.....	....	$x = 17$	.....	.....
(7)	$2y - 6$	$y - 2$	.....	....	$y = 6$	.....	.....
(8)	$3x + 4$	$x - 2$	.....	....	$x = 5$	.....	.....
(9)	$5n - 3$	$4n - 4$	.....	....	$n = 2$	.....	.....
(10)	$8a - 1$	$2a + 4$	.....	....	$a = 8$	.....	.....
(11)	$7b + 2$	$5b + 1$	.....	....	$b = 3$	.....	.....
(12)	$3x + 2y$	$5x - 7y$	.....	....	$x = 4, y = 1$	.....	.....
(13)	$7n - 2s$	$2n + 2s$	.....	....	$n = 3, s = 2$	.....	.....
(14)	$4t + 6u$	$3t + 5u$	.....	....	$t = 10, u = 1$	.....	.....
(15)	$5y + 2z$	$4y - z$	.....	....	$y = 2, z = 4$	.....	.....
(16)	$6a - 4b$	$2a + b$	.....	....	$a = 4, b = 3$	.....	.....
(17)	$8a - 2c$	$3a - c$	.....	....	$a = 3, c = 5$	.....	.....
(18)	$7m + 3n$	$4m + 5n$	.....	....	$m = 5, n = 4$	.....	.....
(19)	$4x + 7y$	$12x - 6y$	.....	....	$x = 2, y = 3$	.....	.....
(20)	$10r - 3s$	$5r + 2s$	.....	....	$r = 6, s = 6$	.....	.....
(21)	$9n - 3l$	$8n - 10l$	.....	....	$n = 2, l = 1$	.....	.....
(22)	$11a + b$	$3a + 2b$	.....	....	$a = 1, b = 2$	.....	.....
(23)	$14k + 3t$	$7k - 2t$	.....	....	$k = 2, t = 2$	.....	.....
(24)	$6x - 5y$	$3x + y$	.....	....	$x = 6, y = 2$	.....	.....
(25)	$15m - 10n$	$5m - 3n$	.....	....	$m = 1, n = 1$	.....	.....
(26)	$5a + 4b$	$a + 3b$	.....	....	$a = 5, b = 4$	.....	.....
(27)	$8x + y$	$4x - y$	.....	....	$x = 2, y = 3$	.....	.....
(28)	$9c - 4d$	$c + 2d$	.....	....	$c = 3, d = 2$	.....	.....
(29)	$5a - 4b$	$4a - 5b$	.....	....	$a = 1.2, b = .3$	.....	.....
(30)	$7x - 5y$	$8x - 12y$	.....	....	$x = 2, y = 1$	.....	.....

(b) Find the numerical values of the perimeter and area of each, using the given lengths for the lines.

7. (a) You have learned previously how to find the product of two binomials as $(2a + 3b)(a + 2b)$, without actually performing the multiplication.

$$(2a + 3b)(a + 2b) = 2a^2 + 7ab + 6b^2.$$

Explain exactly how each term of the product is found.

(b) Multiply $2a - 3b$ by $a - 2b$.

How does this product compare with the one given in 7 (a)?

$$(2a - 3b)(a - 2b) = 2a^2 - 7ab + 6b^2.$$

Why are the first and third terms in both products the same respectively?

Are the second terms of the two products found in the same way?

Why is the second term positive in one problem and negative in the other?

(c) Find by inspection the products of

(1) $(a - b)(a - b)$
(2) $(2a - b)(2a - b)$
(3) $(x - 2y)(x - 2y)$
(4) $(a - 5)(a - 5)$
(5) $(a - 5)(a - 4)$
(6) $(2b - 7)(2b - 3)$
(7) $(3m - 4)(3m - 6)$
(8) $(8r - 3t)(8r - 4t)$
(9) $(5x - 6y)(5x - 3y)$
(10) $(x - y)(x - y)$
(11) $(2x - y)(2x - y)$
(12) $(m - 2n)(m - 2n)$
(13) $(b - 7)(b - 7)$
(14) $(b - 7)(b - 4)$
(15) $(2c - 6)(2c - 4)$
(16) $(3x - 5)(3x - 9)$
(17) $(10t - 5u)(10t - 6u)$
(18) $(4a - 8b)(4a - 12b)$
(19) $(6a - 4c)(6a - 2c)$
(20) $(9x - 3y)(9x - 5y)$

(21) If further practice is needed, make up a set of similar problems and solve your own or those of a classmate.

(d) Multiply $2a - 3b$ by $a + 2b$.

Which term of this product is exactly like the corresponding term in the problem given in 7 (a) and 7 (b)?

Which term is different only in its sign? Why is this sign different?

$$(2a - 3b)\ (a + 2b) = 2a^2 + ab - 6b^2.$$

How did you find the second term in the first product in 7 (a)? The two partial products, $+ 4ab$ and $+ 3ab$, give a sum $+ 7ab$. How did you find the second term in the second product in 7 (b)? In this one, the two partial products, $- 4ab$ and $- 3ab$, give a sum of $- 7ab$. In the third one, in 7 (d), the two partial products, $+ 4ab$ and $- 3ab$, give a sum of $+ ab$.

(e) Find the product of $2a + 3b$ and $a - 2b$

$$(2a + 3b)\ (a - 2b) = 2a^2 - ab - 6b^2.$$

What are the two partial products in this problem? Why is the second term in the complete product negative?

(f) Find by inspection the products of

(1) $(a - 5)\ (a + 4)$	(6) $(n - 6)\ (n + 4)$
(2) $(a + 5)\ (a - 4)$	(7) $(n + 6)\ (n - 4)$
(3) $(2x + 5)\ (3x - 4)$	(8) $(2c + 6)\ (5c - 3)$
(4) $(3x + 2y)\ (2x - 4y)$	(9) $(5r + 3s)\ (2r - 6s)$
(5) $(7b - 3c)\ (4b - 2c)$	(10) $(9m - 3n)\ (3m - 5n)$

(11) $(b - 8)(b + 5)$
(12) $(b + 8)(b - 5)$
(13) $(2x + 7)(5x - 12)$
(14) $(3c + 2d)(4c - 5d)$
(15) $(10a - 2b)(2a - 6b)$
(16) $(x - 8)(x + 7)$
(17) $(x + 8)(x - 7)$
(18) $(2y + 6)(5y - 7)$
(19) $(8a + 3b)(3a - 7b)$
(20) $(10m - 2n)(3m - 5n)$

(21) If further practice is needed, make up a set of similar problems and solve.

(g) The product of any two binomials may be found in this way. Multiply $a + b$ by $a - b$.

$$(a + b)(a - b) = a^2 + 0 - b^2$$
$$= a^2 - b^2$$

The two partial products, $+ ab$ and $- ab$, offset each other, that is, their sum is 0. Therefore, the product may be written as a binomial instead of the usual trinomial.

(h) Find by inspection the products of

(1) $(x + y)(x - y)$
(2) $(m + n)(m - n)$
(3) $(c + 2)(c - 2)$
(4) $(y + 5)(y - 5)$
(5) $(2a + 1)(2a - 1)$
(6) $(2a + 3)(2a - 3)$
(7) $(3x + 7y)(3x - 7y)$
(8) $(x + \frac{1}{2})(x - \frac{1}{2})$
(9) $(2b + \frac{1}{3})(2b - \frac{1}{3})$
(10) $(5a + \frac{1}{2}b)(5a - \frac{1}{2}b)$
(11) $(t + u)(t - u)$
(12) $(r + s)(r - s)$
(13) $(x + 7)(x - 7)$
(14) $(k + 6)(k - 6)$
(15) $(3b + 2)(3b - 2)$
(16) $(7a + 4)(7a - 4)$
(17) $(4a + 6b)(4a - 6b)$
(18) $(a + \frac{1}{4})(a - \frac{1}{4})$
(19) $(3n + \frac{1}{2})(3n - \frac{1}{2})$
(20) $(8x + \frac{1}{3}y)(8x - \frac{1}{3}y)$

(i) (1) The products of two numbers may be found in the same way.

$$16 = 15 + 1 \quad \text{and} \quad 14 = 15 - 1$$

$$\therefore 16 \cdot 14 = (15 + 1)(15 - 1)$$
$$= (15)^2 - (1)^2$$
$$= 225 - 1$$
$$= 224$$

(2) Check by multiplying 16 by 14 in the usual way.

(3) *Another illustration:* $23 \times 17 = ?$

$$\begin{aligned} 23 &= 20 + 3 \\ 17 &= 20 - 3 \\ 23 \cdot 17 &= (20 + 3)(20 - 3) \\ &= 20^2 - 3^2 \\ &= 400 - 9 \\ &= 391 \end{aligned}$$

(4) The method is a quick and easy one to use if the squares of numbers are known or if tables of squares are used. One should know the squares of numbers up to 30. For others use the table of squares given in the Appendix.

(j) Without multiplying find the products of the following numbers:

(1) $15 \cdot 13$	(8) $48 \cdot 52$	(15) $56 \cdot 54$
(2) $18 \cdot 16$	(9) $37 \cdot 33$	(16) $32 \cdot 34$
(3) $21 \cdot 23$	(10) $79 \cdot 81$	(17) $47 \cdot 43$
(4) $26 \cdot 24$	(11) $28 \cdot 22$	(18) $48 \cdot 54$
(5) $29 \cdot 31$	(12) $33 \cdot 27$	(19) $63 \cdot 57$
(6) $28 \cdot 26$	(13) $17 \cdot 13$	(20) $87 \cdot 93$
(7) $18 \cdot 12$	(14) $21 \cdot 19$	(21) $83 \cdot 87$

(k) (1) As you learned to find the product of $(a + b)(a + b)$, so you may find the square of any number.

$$52 = 50 + 2$$

$$\begin{aligned} \overline{52}^2 &= (50 + 2)(50 + 2) \\ &= 2500 + 200 + 4 \\ &= 2704 \end{aligned}$$

(2) It is easier to put some numbers in the form of $(a - b)(a - b)$; as, 59 equals $50 + 9$, but it also equals $60 - 1$.

Which form is more convenient, $59 = 50 + 9$ or $59 = 60 - 1$?

Since $59 = 60 - 1$

$$\begin{aligned} \overline{59}^2 &= (60 - 1)(60 - 1) \\ &= 3600 - 120 + 1 \\ &= 3481 \end{aligned}$$

(3) In the seventh grade you learned a short method of squaring a number ending in $\frac{1}{2}$, .5, or 5; as,

$$\begin{aligned} (4\tfrac{1}{2})^2 &= 4 \cdot 5 + \tfrac{1}{4} \\ &= 20\tfrac{1}{4} \\ (35)^2 &= 3 \cdot 4 \text{ with 25 annexed} \\ &= 1225 \end{aligned}$$

If $4\frac{1}{2}$ is written as a binomial the reason for this short method will be clear.

$$\begin{aligned} 4\tfrac{1}{2} &= 4 + \tfrac{1}{2} \\ (4\tfrac{1}{2})^2 &= (4 + \tfrac{1}{2})(4 + \tfrac{1}{2}) \\ &= 4 \cdot 4 + 4 + \tfrac{1}{4} \\ &= 5 \cdot 4 + \tfrac{1}{4} \\ &= 20\tfrac{1}{4} \\ (35)^2 &= (30 + 5)(30 + 5) \\ &= 30 \cdot 30 + 10 \cdot 30 + 25 \\ &= 40 \cdot 30 + 25 \\ &= 1200 + 25 \\ &= 1225 \end{aligned}$$

(4) 36 may be written as $30 + 6$, $40 - 4$, or $35 + 1$. Which form can be most easily squared?

$$\begin{aligned} (36)^2 &= (35 + 1)(35 + 1) \\ &= 35^2 + 2 \cdot 35 + 1 \\ &= 1225 + 70 + 1 \\ &= 1296 \end{aligned}$$

(5) Multiply 36 by 34 in the shortest way.

$$\begin{aligned} 36 \times 34 &= (35 + 1)(35 - 1) \\ &= 1225 - 1 \\ &= 1224 \end{aligned}$$

(l) Write the following results without actual multiplication:

(1) $(61)^2 =$
(2) $(49)^2 =$
(3) $39 \times 41 =$
(4) $54 \times 56 =$
(5) $(71)^2 =$
(6) $(69)^2 =$
(7) $49 \times 51 =$
(8) $74 \times 76 =$
(9) $(46)^2 =$
(10) $(48)^2 =$
(11) $(77)^2 =$
(12) $89 \times 91 =$
(13) $46 \times 44 =$
(14) $(56)^2 =$
(15) $(89)^2 =$
(16) $59 \times 61 =$
(17) $(91)^2 =$
(18) $66 \cdot 64 =$
(19) $97 \cdot 103 =$
(20) $69 \cdot 71 =$

V. Finding Dimensions from Areas — Factoring

1. (a) You have already learned how to find the dimensions of a rectangle from a given area, such as $2a^2 + 7ab + 6b^2$. To find these dimensions you had to find the two factors whose product is the given area.

(b) What is a possible set of factors that make $2a^2$? Is there more than one set possible for this term?

(c) What are the possible sets of factors for $6b^2$?

(d) Try these different sets together to see which ones give the correct middle term, $+ 7ab$.

$$\begin{array}{c} 2a + b \\ a + 6b \\ \hline \end{array} \qquad\qquad \begin{array}{c} 2a + 3b \\ a - 2b \\ \hline \end{array}$$

(e) Why is it not necessary to try such combinations as these?

$$\begin{array}{r} 2a + 6b \\ a + \ b \\ \hline \end{array} \qquad \begin{array}{r} 2a + 2b \\ a + 3b \\ \hline \end{array}$$

(f) From these trials we find that,

$$2a^2 + 7ab + 6b^2 = (2a + 3b)(a + 2b)$$

(g) Since all of the signs in the trinomial are plus, we need pay little attention to signs, for we know that the signs in the two binomial factors must be also plus.

(h) Find the factors of the following:

(1) $a^2 + 4ab + 4b^2$	(5) $15a^2 + 34ab + 15b^2$
(2) $15x^2 + 11xy + 2y^2$	(6) $14x^2 + 41xy + 15y^2$
(3) $6b^2 + 19bc + 8c^2$	(7) $12t^2 + 10tu + 2u^2$
(4) $12an^2 + 26an + 12a$	(8) $8x^2 + 23xy + 14y^2$

NOTE. — Whenever possible, always take out the common monomial factor, *first*.

2. (a) If the given area of the rectangle is $2a^2 - 7ab + 6b^2$, we must think about the signs, as well as the terms of the factors, when we find its dimensions.

(b) Since the sign of the third term, $+ 6b^2$, is positive, we know that its two factors have *like* signs, either both plus or both minus.

(c) If both were plus, what sign would $7ab$ have?
If both were minus, what sign would $7ab$ have?
Which ones are correct?

$$2a^2 - 7ab + 6b^2 = (2a - 3b)(a - b)$$

(d) Find the factors of the following:

(1) $3x^2 - 7xy + 4y^2$	(4) $12r^2 - 15rs + 3s^2$
(2) $5a^2 - 8ab + 3b^2$	(5) $3m^2 - 5mn + 2n^2$
(3) $10c^2 - 25cd + 15d^2$	(6) $10t^2 - 11tu + 3u^2$

(7) $12a^2 - 16ax + 5x^2$
(8) $15x^2 + 26xy + 8y^2$
(9) $21a^2 - 13ab + 2b^2$
(10) $18c^2 - 27cd + 10d^2$
(11) $28r^2 - 41rs + 15s^2$
(12) $m^2 - 14mn + 48n^2$
(13) $2t^2 - 24tu + 70u^2$
(14) $6a^2 - 13ax + 6x^2$
(15) $25x^2 - 60xy + 36y^2$
(16) $16a^2 + 30ab + 9b^2$
(17) $20c^2x - 56cdx + 15d^2x$
(18) $12r^2 - 37rs + 21s^2$
(19) $18m^2 - 27mn + 10n^2$
(20) $24a^2 - 31ab + 10b^2$

3. (a) If the given area of the rectangle is

$$2a^2 + ab - 6b^2 \text{ or } 2a^2 - ab - 6b^2,$$

to find the factors, or dimensions, we must think still more about the signs.

(b) Whenever the third term of the trinomial has a minus sign, we know that one binomial factor has a plus sign and one has a minus sign. Why?

(c) This means that one of the cross products will be positive and the other negative. How must you combine their coefficients to find the middle term of the trinomial?

(d) $2a^2 + ab - 6b^2$ | $2a^2 - ab - 6b^2$

The possible factors of both of these trinomials are

$2a - 3b$	$2a + 3b$	$2a + b$	$2a - b$
$a + 2b$	$a + 2b$	$a - 6b$	$a + 6b$
$-3ab$	$+3ab$	$+ab$	$-ab$
$+4ab$	$-4ab$	$-12ab$	$+12ab$
$+ab$	$-ab$	$-11ab$	$+11ab$

Evidently the third and fourth sets are not correct. The first set gives the correct middle term of the first trinomial and the second set gives the correct middle term of the second trinomial. Therefore, $2a^2 + ab - 6b^2 = (2a - 3b)(a + 2b)$ and $2a^2 - ab - 6b^2 = (2a + 3b)(a - 2b)$

(e) Test the factoring by giving some numerical values to a and b. If both sides of the equation have the same value, the factors must be correct.

(1) Let $a = 2$ and $b = 3$

$$
\begin{aligned}
2a^2 + ab - 6b^2 &= (2a - 3b)(a + 2b) \\
2\cdot2^2 + 2\cdot3 - 6\cdot3^2 &= (2\cdot2 - 3\cdot3)(2 + 2\cdot3) \\
8 + 6 - 54 &= (4 - 9)(2 + 6) \\
-40 &= (-5)(8) \\
-40 &= -40
\end{aligned}
$$

(2)

$$
\begin{aligned}
2a^2 - ab - 6b^2 &= (2a + 3b)(a - 2b) \\
2\cdot2^2 - 2\cdot3 - 6\cdot3^2 &= (2\cdot2 + 3\cdot3)(2 - 2\cdot3) \\
8 - 6 - 54 &= (4 + 9)(2 - 6) \\
-52 &= (13)(-4) \\
-52 &= -52
\end{aligned}
$$

(f) Find the factors of the following and check:

(1) $3x^2 + 2xy - 8y^2$
(2) $3a^2 - 2ab - 8b^2$
(3) $18am^2 + 3amn - 6an^2$
(4) $20c^2 - 6cd - 2d^2$
(5) $12k^2 - 4lk - 5l^2$
(6) $28t^2 + 2tu - 6u^2$
(7) $6a^2 + 11ax - 10x^2$
(8) $14t^2 - 29tu - 15u^2$
(9) $18bx^2 + 21bxy - 4by^2$
(10) $16m^2 - 2mn - 3n^2$
(11) $12c^2 + 19cd - 21d^2$
(12) $18y^2 - 19yz - 12z^2$
(13) $6t^2 + 5tu - 4u^2$
(14) $20x^2 - 7xy - 6y^2$
(15) $40m^2 + mn - 6n^2$
(16) $10c^2 - 7cd - 12d^2$
(17) $28a^2 + 23ay - 15y^2$
(18) $t^2 - 5tu - 24u^2$
(19) $27x^2 + 30xy - 8y^2$
(20) $10a^2 - 7ab - 6b^2$

4. (a) If $a + b$ is multiplied by $a - b$, the product is really a trinomial with zero as a second term; as, $a^2 + 0 - b^2$. The product is usually written as the binomial $a^2 - b^2$.

(b) To find the factors of $a^2 - b^2$, we may think of it as a trinomial $a^2 + 0 - b^2$, and factor it in the same way as we factored the other trinomials.

The factors of a^2 are a and a; of $-b^2$ are $+b$ and $-b$. Their cross product is 0.

Therefore, $a^2 - b^2 = (a + b)(a - b)$.

(c) Factor $25x^2 - 16y^2$.

$$\begin{array}{r} 5x + 4y \\ \underline{5x - 4y} \\ +20xy \\ \underline{-20xy} \\ 0 \end{array}$$

Therefore, $25x^2 - 16y^2 = (5x + 4y)(5x - 4y)$.

(d) Test when $x = 4$ and $y = 3$.

$$\begin{aligned} 25x^2 - 16y^2 &= (5x + 4y)(5x - 4y) \\ 25 \cdot 4^2 - 16 \cdot 3^2 &= (5 \cdot 4 + 4 \cdot 3)(5 \cdot 4 - 4 \cdot 3) \\ 25 \cdot 16 - 16 \cdot 9 &= (20 + 12)(20 - 12) \\ 400 - 144 &= (32)(8) \\ 256 &= 256 \end{aligned}$$

(e) Find the factors of the following and check:

(1) $9x^2 - 4y^2$
(2) $25a^2 - 49b^2$
(3) $9x^2 - 16y^2$
(4) $64a^2 - 9b^2$
(5) $49c^2 - 9d^2$
(6) $\frac{1}{4}r^2 - s^2$
(7) $m^2 - 81n^2$
(8) $100by^2 - 36bz$
(9) $32c^2 - 50b^2$
(10) $25h^2 - 9a^2$
(11) $75h^2 - 48b^2$
(12) $\frac{16}{9}cx^2 - \frac{1}{9}cy^2$
(13) $400a^2 - 9b^2$
(14) $36c^2 - 49d^2$
(15) $64r^2 - 25s^2$
(16) $225t^2 - 121u^2$
(17) $361y^2 - 289z^2$
(18) $\frac{5}{9}a^2 - \frac{5}{16}b^2$
(19) $144m^2 - 121n^2$
(20) $81c^2 - 25d^2$

5. (a) Test your achievement by factoring the examples given below.

(b) Keep a record of the time taken.

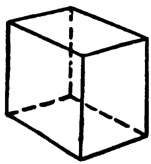
Factor these problems again every second day until you can do them quickly and accurately.

(1) $a^2x + 5abx + 6b^2x$
(2) $4x^2 - 20xy + 25y^2$
(3) $12m^2 + 5mn - 2n^2$
(4) $15t^2 - 2tu - 8u^2$
(5) $98a^2 - 8b^2$
(6) $20r^2 + 31rs + 12s^2$
(7) $15x^2 + 14xy - 8y^2$
(8) $81c^2 - 16d^2$
(9) $6a^2 - 16ab + 8b^2$
(10) $28t^2 - 9tu - 9u^2$
(11) $20r^2 - 43rs + 21s^2$
(12) $12x^2 + 2xy - 80y^2$
(13) $40c^2 - 38cd - 15d^2$
(14) $100a^2 - 81b^2$
(15) $15t^2 + 28tu + 12u^2$
(16) $144k^2 - 169b^2$
(17) $12a^2 - 8ab - 15b^2$
(18) $28x^2 + 25xy - 8y^2$
(19) $20m^2 - 53mn + 18n^2$
(20) $40t^2 + 21tu + 2u^2$
(21) $99c^2 - 15cd - 6d^2$
(22) $28x^2 - 41xy + 15y^2$
(23) $24m^2 + 10mn - n^2$
(24) $18t^2 + 19tu + 5u^2$
(25) $\frac{3}{25}a^2 - \frac{3}{16}b^2$
(26) $35x^2 + 11xy - 6y^2$
(27) $20m^2 - 13m - 21$
(28) $225t^2 - 36u^2$
(29) $12a^2 - 16ab - 35b^2$
(30) $9c^2 + 11cd^2 + 2d^2$
(31) $2x^2 + 2xy - 84y^2$
(32) $60m^2 - 20mn - 25n^2$
(33) $12t^2 + 36tu + 15u^2$
(34) $16a^2 - 40ab + 25b^2$
(35) $25c^2 - 81d^2$
(36) $48x^2 + 26xy + 3y^2$
(37) $21m^2 + 13mn - 18n^2$
(38) $64a^2 - 49x^2$
(39) $27b^2 - 3by - 14y^2$
(40) $50a^2 - 25ab - 12b^2$

(c) Check the factors of each by giving convenient numerical values to the letters.

VI. Finding Volume or the Product of Three Factors

1. An edge of a cube is $2a + 3b$ inches.
(a) What is the area of its base?
(b) How many cubic inches does it contain?
(c) *Solution of* (a):



$$S_\square = e^2$$
$$= (2a + 3b)^2$$
$$= (2a + 3b)(2a + 3b)$$
$$= 4a^2 + 12ab + 9b^2$$

(d) *Solution of* (b):

$$
\begin{aligned}
V_{cu} &= e^3 \\
&= (2a+3b)^3 \\
&= (2a+3b)(2a+3b)(2a+3b) \\
&= (4a^2+12ab+9b^2)(2a+3b) \\
&= 8a^3+36a^2b+54ab^2+27b^3
\end{aligned}
$$

Process

$$
\begin{array}{l}
4a^2 + 12ab + 9b^2 \\
\underline{2a + 3b \qquad\qquad\qquad} \\
8a^3 + 24a^2b + 18ab^2 \\
\underline{\qquad + 12a^2b + 36ab^2 + 27b^3} \\
8a^3 + 36a^2b + 54ab^2 + 27b^3
\end{array}
$$

(e) Find the number of cubic inches in the volume when a is 4 in. long and b is 3 in. long.

$V_{cu} = 8a^3 + 36a^2b + 54ab^2 + 27b^3$

$= 8(4)^3 + 36(4)^2(3) + 54(4)(3)^2 + 27(3)^3$

Raising numbers to powers

$= 8(64) + 36(16)(3) + 54(4)(9) + 27(27)$

Multiplying factors

$= 512 + 1728 + 1944 + 729$

Adding terms

$= 4913$ cu. in.

(f) Check first by finding the numerical length of the edge.

$$
\begin{aligned}
2a + 3b &= 2(4) + 3(3) \\
&= 8 + 9 \\
&= 17 \\
V_{cu} &= e^3 \\
&= 17^3 \\
&= 17^2 \cdot 17 \\
&= 289 \cdot 17 \\
&= 4913 \text{ cu. in.}
\end{aligned}
$$

Process

$$
\begin{array}{r}
289 \\
\underline{17} \\
2023 \\
\underline{289} \\
4913
\end{array}
$$

(g) (1) Explain why $4a$ times $2a$ gives $8a^2$.

What must you do with the coefficients of the factors to get the coefficients of the product?

(2) What must you do with the exponents of the like letters of the factors to get the exponent of the same letter in the product?

(3) Explain how $12ab$ multiplied by $2a$ gives a product of $24a^2b$.

(4) Explain how each partial product is found.

(5) Why do we place the $12a^2b$ under the $24a^2b$ in the multiplication?

(6) Why must we not place $36ab^2$ under the $24a^2b$?

(7) In order to combine two terms into one, that is, in order to add two terms, what must be true of the letters in the terms and their exponents?

(8) How does a^2b differ from ab^2?
Show their numerical difference when a is 4 and b is 3.

(h) (1) The second equation in (e) says that:
$V_{cu} = 8(4)^3 + 36(4)^2 (3) + 54(4) (3)^2 + 27(3)^3$

(2) This equation tells us to do a number of things. In the first term it tells us to cube 4 and multiply the result by 8.

(3) Why must we not multiply by 8 before we cube the number?
What number value would we get for this term if we multiply by 8 before we raise the 4 to the third power? How does this value compare with the true value of this term?

(4) The second term is $36(4)^2(3)$. What must be done first in the term?
After the 4 is squared we have 36(16) (3) or $36 \cdot 16 \cdot 3$.

Does it make any difference in which order we multiply these factors?

(5) After each term is simplified, that is, after each term is changed to a single number, what must be done with all the terms?

(6) It is very necessary to do the work in the correct order.

First, raise each factor to the power indicated.
Second, multiply the factors of each term.
Last, add the terms.

2. (a) Find the area of the base and the volume of each cube from the given size of edges.

(b) From the numerical lengths given to each letter in the edge, find the numerical value of the area of the base and the volume of each cube.

(c) Check numerical values by finding the length of the edge first.

No.	Edge	Area of Base	Volume	Numerical Value of		
				Edge	Area of Base	Volume
(1)	$a + b$	....	....	$a = 2, b = 3$	....	....
(2)	$x - y$	....	....	$x = 7, y = 3$	....	....
(3)	$n + 1$	....	....	$n = 6$	....	....
(4)	$n - 1$	....	....	$n = 9$	....	....
(5)	$2a - 3b$	....	....	$a = 7, b = 2$	....	....
(6)	$m + n$	....	....	$m = 10, n = 2$	....	....
(7)	$t - n$	....	....	$t = 9, u = 3$	....	....
(8)	$a + 2$	....	....	$a = 12$	....	....
(9)	$a - 2$	....	....	$a = 10$	....	....
(10)	$2b + 3d$	....	....	$b = 3, d = 4$	....	....
(11)	$3x - 4y$	....	....	$x = 8, y = 3$	....	....
(12)	$4a - b$	....	....	$a = 6, b = 3$	....	....
(13)	$5a - 2x$	....	....	$a = 4, x = 2$	....	....
(14)	$6b + y$	....	....	$b = 5, y = 7$	....	....
(15)	$4m - n$	....	....	$m = 6, n = 4$	....	....
(16)	$c + 2d$	....	....	$c = 16, d = 6$	....	....
(17)	$x - 3y$	....	....	$x = 15, y = 2$	....	....
(18)	$8x + 5y$	....	....	$x = 1, y = 2$	....	....
(19)	$7a - 2b$	....	....	$a = 3, b = 3$	....	....
(20)	$10n + 6$	....	....	$n = 1$	....	....

3. (a) What is the formula for the volume of any rectangular solid?

(b) Find the volume of a rectangular bin that has the following dimensions:

$$\begin{aligned} \text{Length} &= 2x + 3y \\ \text{Width} &= x + 2y \\ \text{Height} &= 2x - y \end{aligned}$$

(c) *Solution:*

	Process
$V_{pr} = lwh$	
$= (2x + 3y)(x + 2y)(2x - y)$	$2x^2 + 7xy + 6y^2$
$= (2x^2 + 7xy + 6y^2)(2x - y)$	$2x - y$
	$4x^3 + 14x^2y + 12xy^2$
$= 4x^3 + 12x^2y + 5xy^2 - 6y^3$	$- 2x^2y - 7xy^2 - 6y^3$
	$4x^3 + 12x^2y + 5xy^2 - 6y^3$

(d) Find the number of cubic feet in the volume when $x = 3$ ft. and $y = 2$ ft.

Check your evaluation.

	Check
$V = 4x^3 + 12x^2y + 5xy^2 - 6y^3$	
$= 4 \cdot 3^3 + 12 \cdot 3^2 \cdot 2 + 5 \cdot 3 \cdot 2^2 - 6 \cdot 2^3$	$l = 2x + 3y = 6 + 6 = 12$
$= 4 \cdot 27 + 12 \cdot 9 \cdot 2 + 5 \cdot 3 \cdot 4 - 6 \cdot 8$	$w = x + 2y = 3 + 4 = 7$
$= 108 + 216 + 60 - 48$	$h = 2x - y = 6 - 2 = 4$
	$lwh = 12 \cdot 7 \cdot 4$
$= 336$ cu. ft.	$= 336$ cu. ft.

4. (a) From the given dimensions of rectangular solids, find the volume of each.

(b) From the numerical values given to the letters in the dimensions, find the numerical measure of each volume.

(c) Check each.

No.	Length	Width	Height	Volume	Numerical Value of	
					Lines	Volume
(1)	$3x + 2y$	$x + y$	$3x - y$	....	$x = 2,\ y = 1$	....
(2)	$4a + 3b$	$2a + b$	$3a - 2b$	....	$a = 3,\ b = 2$	....
(3)	$5n + 8$	$3n + 4$	$4n - 3$	....	$n = 4$	....
(4)	$6c + 5d$	$8c - 3d$	$5c - 2d$	....	$c = 4,\ d = 3$	....
(5)	$7t - 2n$	$6t - n$	$9t - 8n$	....	$t = 5,\ n = 4$	....
(6)	$8a - 3x$	$3a + x$	$5a - 3x$	....	$a = 5,\ x = 3$	....
(7)	$7b - 2y$	$5b - 3y$	$3b + 2y$	....	$b = 6,\ y = 2$	....
(8)	$9m - 2n$	$7m + n$	$3m + n$	....	$m = 7,\ n = 3$	....
(9)	$8x - y$	$4x + 3y$	$3x + 2y$	....	$x = 8,\ y = 5$	....
(10)	$10a + 3b$	$6a + b$	$4a + 3b$	....	$a = 3,\ b = 4$	....
(11)	$12n - 6$	$7n + 8$	$5n + 7$	....	$n = 5$	....
(12)	$7c + 2d$	$3c + 4d$	$8c - 3d$	....	$c = 5,\ d = 7$	....
(13)	$5t + 3u$	$8t - 7u$	$4t - u$	....	$t = 9,\ u = 5$	....
(14)	$6a + 5x$	$9a - 7x$	$5a + 4x$	....	$a = 3,\ x = 2$	....
(15)	$b + 6y$	$5b - 2y$	$7b - 3y$	....	$b = 3,\ y = 3$	....
(16)	$9m - 2n$	$7m - 4n$	$6m - 5n$	....	$m = 6,\ n = 4$	....
(17)	$8x + 7y$	$6x - 5y$	$5x - 2y$	....	$x = 8,\ y = 3$	....
(18)	$10a + 7b$	$5a - 2b$	$7a - 6b$	....	$a = 6,\ b = 3$	....
(19)	$12n - 7$	$3n + 6$	$8n - 5$	....	$n = 5$	....
(20)	$4c + 3d$	$7c - d$	$8c - 5d$	....	$c = 7,\ d = 6$	....
(21)	$11a + c$	$4a + 3c$	$7a - 2c$	....	$a = 3,\ c = 5$	....
(22)	$10x - y$	$4x + 5y$	$8x - 3y$	....	$x = 4,\ y = 3$	....

CHAPTER TWO

MEANING OF ALGEBRAIC EXPRESSIONS

A. DRAWING GEOMETRIC FIGURES TO REPRESENT ALGEBRAIC EXPRESSIONS

1. (a) Draw the geometric figures that might represent the following algebraic expressions.

(1) $n + 4$
(2) $a - b$
(3) n^2
(4) $n(n - 2)$
(5) $a + b + c$
(6) $4x^2 + 4xy + y^2$
(7) $(x - y)^2$
(8) $(m + 1)(m + 1)(m + 1)$
(9) $(m + 1)(m + 2)(m + 3)$
(10) $a^2 + b^2 + c^2 + 2ab + 2ac + 2bc$
(11) $(2m + 5n)^3$
(12) $a^3 + 3a^2b + 3ab^2 + b^3$
(13) $(2x - 5)^2$
(14) $(2x + 5)(2x - 5)$
(15) πr^2
(16) xy

B. DEGREE OF ALGEBRAIC EXPRESSIONS

I. First Degree

Such algebraic expressions as n, $2a - b$, $a + b + c$, etc., which may be represented geometrically by lines are *first degree expressions.* In a first degree expression no term may contain more than one letter factor. Which power of the literal or letter factor is used?

II. Second Degree

Such algebraic expressions as x^2, $a^2 - 4$, $m^2 + 2mn + n^2$, xy, $a^2 + a - 6$, etc., which may be represented by areas of geometric figures, are *second degree expressions.* In a second degree expression, at least one term must have two literal factors. If these two factors are alike, the term will have the second power of the letter, as x^2. If the two literal factors are unlike, the term will be the product of the first powers of the letters, as xy or $5ab$. A second degree expression is also called a *quadratic expression.* Since *quadratic* comes from a Latin word which means *square,* why is it an appropriate name for an expression of the second degree?

III. Third Degree

Such algebraic expressions as x^3, a^2b, mn^2, abc, $a^3 - 1$, $x^3 + 3x^2 + 3x + 1$, etc., which may be represented by the volumes of geometric figures are *third degree expressions.*

In a third degree expression how many literal factors must be in at least one term? When may a third degree expression contain only the first powers of the letters? When the second power? When the third power?

Third degree expressions are sometimes called *cubic expressions.* Why is this name appropriate?

(a) Give the degree of each of the following expressions.

(b) Give your reason for each.

(1) $x + 6$
(2) $n^2 + 7n + 12$
(3) $x^3 + x^2y$
(4) $(5x - 7)\ (3x + 6)$
(5) $(3t + 4u)^2$
(6) $8x^3 + 36x^2y + 54xy^2 + 27y^3$
(7) $a + x + y$
(8) $27x^3$
(9) $(r + s)\ (2r + 7s)\ (3r - 4s)$

(10) $x^2 + y^2 + z^2 - 2xy - 2xz + 2yz$
(11) $2a(a + 2b)$
(12) $x + 4y - 7z$
(13) $(4c - 3d)(4c + 3d)$
(14) $x^3 + 3x^2y + 3xy^2 + y^3$
(15) $49t^2 + 42tu + 9u^2$
(16) $(x + 2)(x + 3)(x + 4)$

C. MEANINGS OF ALGEBRAIC EXPRESSIONS OTHER THAN GEOMETRIC

1. We live in a world which has only three dimensions. We find it very difficult to imagine a world of four dimensions, but there are some people with imaginations sufficiently vivid to do so.

2. If algebraic expressions were used only to measure geometric figures, we would need none higher than the third degree. But letters or algebraic expressions may be used to represent many other things beside measurements of lines. For example:

(a) n may stand for any number, as the number of cents or dollars an article costs; or the number of miles one walks, rides, or rows in an hour; or the number of pounds some one or something weighs. These are a few of many uses.

(b) You have found that the amount of interest one pays changes with the principal, the rate, or the time; that is, I = PRT. This formula is true always, although each letter may have a great many numerical values. This formula, as well as all of those used in mensuration, really gives a general rule for solving any similar problem. Because these letters give *general* rules which may be used in solving *particular* problems by giving different number values to the letters, algebraic numbers or letters are

sometimes called *general numbers*. To illustrate, in the formula for the circumference of a circle, $2\pi r$, the r stands for all radii, in general, and each particular circumference may be found by substituting the definite number which measures the radius of a particular circle.

I. Solving Problems by Formulas

1. The distance an automobile travels depends upon two factors which may vary in each particular trip; but in general the distance always depends upon the rate, or number of miles traveled in an hour, and the time, or the number of hours it takes to make the trip.

If d = distance, r = rate, and t = time, then, $d = rt$ or $rt = d$.

This equation gives a general solution for all such problems about distance, rate, and time.

(a) Find the distance when the rate is 25 miles an hour and the time is $3\frac{1}{2}$ hours.

(b) Find the distance if r = 16 ft. a second and t = 18 seconds.

2. (a) Find the time it takes a train to run 231 miles if the rate averages 35 miles an hour.

(b) *Solution:*

Given $r = 35$

$d = 231$

To find t

Formula, $rt = d$

$35t = 231$

Dividing both sides by 35,

$$\frac{\cancel{35t}}{\cancel{35}} = \frac{\overset{33}{\cancel{231}}}{\underset{5}{\cancel{35}}}$$

$t = 6\frac{3}{5}$ hr.

$= 6$ hr. 36 min.

3. Find the time it takes a train to run 231 miles at 33 miles an hour.

4. A time table showing trains from Buffalo to Chicago gives Toledo 296 miles from Buffalo and Chicago 540 miles from Buffalo. What is the average rate of a train which leaves Toledo at 10.05 A.M. and arrives in Chicago at 5 P.M.?

5. (a) If d stands for the number of dollars you put in the savings bank each week, what general number would represent the number of dollars saved each month? Each year?
   (b) Write an equation which says that the number of dollars saved each month is 16.
   (c) From this equation find the number of dollars saved each week.

6. Make and solve a problem of your own about the number of dollars saved each year.

II. Translation of English Statements into Algebraic Expressions

1. (a) The length of a rectangular field is three times its width. If n is the number of rods in the width, what may represent the number of rods in the length?
   (b) What expression may represent the number of rods in the perimeter?
   (c) Make an equation which says the number of rods in the perimeter is 256.
   (d) Solve this equation to find the length and width of the field.
   (e) Find these dimensions in feet.

2. (a) The length of a field is 6 rd. more than twice its width.
   (b) Which is the shorter dimension?
   (c) It is usually more convenient to let some letter stand for the smaller of two numbers.

(d) If n equals the number of rods in the width, what expression may represent the number of rods in the length?

(e) What expression may represent the perimeter?

(f) If the perimeter is 192 rd., find the two dimensions.

(g) *Solution:*

Let n = no. of rd. in width.

Then $2n + 6$ = no. of rd. in length.

$$\begin{aligned} P_{\square} &= 2(l + w) \\ &= 2(2n + 6 + n) \\ &= 2(3n + 6) \\ &= 6n + 12 \end{aligned}$$

But the problem states that the perimeter equals 192 rd.

Since the two expressions $6n + 12$ and 192 both represent the same perimeter, these two expressions must be equal to each other.

$$\begin{array}{rcll} P_{\square} &=& 6n + 12 & \\ P_{\square} &=& 192 & \\ \therefore\ 6n + 12 &=& 192 & \text{Equality Axiom} \\ +\,12 &=& 12 & \text{Identity Axiom} \\ \hline 6n &=& 180 & \text{Subtraction Axiom} \end{array}$$

$$\frac{\cancel{6}n}{\cancel{6}} = \frac{\overset{30}{\cancel{180}}}{\cancel{6}} \qquad \text{Division Axiom}$$

$$\begin{aligned} n &= 30 \text{ rd. in width} \\ 2n + 6 &= 66 \text{ rd. in length} \end{aligned}$$

Check. $P_{\square} = 2(2n + 6 + n) = 192$

$$\begin{aligned} 2(60 + 6 + 30) &= 192 \\ 2 \cdot 96 &= 192 \\ 192 &= 192 \end{aligned}$$

(h) In the solution of the equation, we referred to several axioms. An axiom is a statement which one is willing to assume to be true without

proof. Already we have used several axioms. Among them are:

(1) The *Equality Axiom.* If two or more expressions are equal to the same thing or to equal things, they are equal to each other.

(2) The *Identity Axiom.* Any number or anything is always equal to itself. That is, the two things are identically the same.

(3) *Addition Axiom.* If equals are added to equals, the sums are equal.

Applied to an equation, this axiom means that a number may be added to one member of an equation if the same number or an equal number is added to the second member. The equation will still be in balance. Give an illustration.

(4) *Subtraction Axiom.* If equals are subtracted from equals, the remainders are equal.

Explain what this axiom means when applied to an equation. Illustrate it.

(5) *Multiplication Axiom.* If equals are multiplied by equals, the products are equal.

Explain and illustrate this axiom when applied to equations.

(6) *Division Axiom.* If equals are divided by equals (not zeros) the quotients are equal.

Explain and illustrate this axiom applied to equations.

(i) We may do a great many things to equations, but we must always be careful to do the same thing to both members.

(j) Why was it necessary to subtract 12 from each member of the equation in 2 (g)?

3. (a) The length of a picture frame is 3 in. less than 2 times the width.

(b) Let some letter represent the number of inches in the width. What expression may represent the number of inches in the length?
(c) If the perimeter is 84 in., find the dimensions.
(d) Find an expression to represent the perimeter.
(e) Make the equation and solve it.
(f) Check results.

4. Translate the following English statements into algebraic expressions:
(a) Five times a number increased by 4.

$$5 \quad \cdot \quad n \quad + \quad 4 \text{ or } 5n + 4.$$

(b) A number decreased by 8.
(c) Six times a number decreased by the number.
(d) 36 exceeds 20.
(e) 36 exceeds x.
(f) $x + y$ exceeds y.
(g) Four times a number decreased by 1 exceeds three times the number.
(h) A number greater than n by 2.
(i) A number less than x by 1.
(j) The sum of a number, twice the number, and four times the number.
(k) The larger part of a yard if the smaller is 10.
(l) The larger part of a yard if the smaller is a.
(m) The larger part of x if the smaller is b.
(n) The sum of two boys' ages if one is 5 years older than the other.
(o) Three consecutive numbers.
(p) Three consecutive even numbers.
(q) Three consecutive odd numbers.
(r) The sum of two consecutive numbers.
(s) The difference between two consecutive numbers.
(t) The product of two numbers.
(u) Three-fourths of a number.

(v) Four-fifths of a number plus 7.
(w) The excess of 12 over 10.
(x) The excess of 12 over x.
(y) The excess of n over 4.
(z) The excess of twice a number over 3.

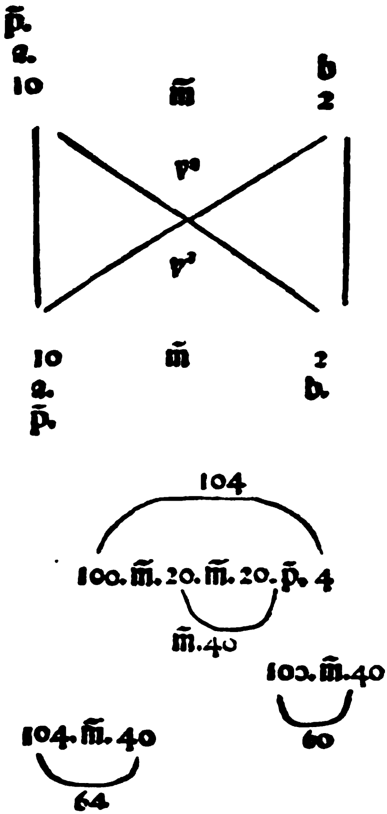
D. HOW ALGEBRAIC SYMBOLS GREW

I. Pacciuolo

You have found that a long English sentence can be briefly expressed in an algebraic equation. But 300 years ago few of our symbols were known. How, then, did the mathematicians make their calculations?

If you examine an algebra written in the fifteenth or sixteenth century, you will find that most of the pages were of solid print, and the discussions rhetorical, chiefly in Latin, Italian, or French. The few symbols used seem very cumbersome.

We find it very easy to multiply $10a - 2b$ by $10a - 2b$. Compare ours with the way an Italian monk, named Pacciuolo or Pacioli, multiplied them as shown in his book published in 1494.



By Bombelli

$$\overset{\overset{1}{\smile}}{6} \text{ p. } 4.$$
$$\overset{\overset{1}{\smile}}{5} \text{ p. } 6.$$

$$\overset{\overset{1}{\smile}}{11} \text{ p. } 10.$$

$$\overset{\overset{2}{\smile}}{12}.\text{m.}\overset{\overset{1}{\smile}}{6}.\text{p.}4.$$
$$\overset{\overset{2}{\smile}}{5}. \text{ p. } \overset{\overset{1}{\smile}}{9}.\text{m.}5.$$

$$\overset{\overset{2}{\smile}}{17}. \text{ p. } \overset{\overset{1}{\smile}}{3}.\text{m.}1.$$

$$\overset{\overset{2}{\smile}}{3}. \text{ p. } \overset{\overset{1}{\smile}}{4}. \text{ m. } 2$$
$$\overset{\overset{1}{\smile}}{4}. \text{ p. } 2$$

$$\overset{\overset{3}{\smile}}{12}. \text{ p. } \overset{\overset{2}{\smile}}{16}. \text{ m. } \overset{\overset{1}{\smile}}{8}. \text{ p. } \overset{\overset{2}{\smile}}{6}. \text{ p. } \overset{\overset{1}{\smile}}{8}. \text{ m. } 4.$$

$$\overset{\overset{3}{\smile}}{12}. \text{ p. } \overset{\overset{2}{\smile}}{22}. \text{ m. } 4.$$

MODERN

In modern symbols the first problem is

$$\begin{array}{r} 6x + 4 \\ 5x + 6 \\ \hline 11x + 10 \end{array}$$

The second is

$$\begin{array}{r} 12x^2 - 6x + 4 \\ 5x^2 + 9x - 5 \\ \hline 17x^2 + 3x - 1 \end{array}$$

The third is

$$\begin{array}{l} \quad\ 3x^2 + 4x - 2 \\ \qquad\qquad\ 4x + 2 \\ \hline 12x^3 + 16x^2 - 8x + 6x^2 + 8x - 4 \end{array}$$

This product combines to

$$12x^3 + 22x^2 - 4$$

II. Bombelli

The earliest approach to the use of exponents was in 1572 by Rafael Bombelli, who wrote $\overset{1}{\smile}$ for an unknown quantity, $\overset{2}{\smile}$ for its square, $\overset{3}{\smile}$ for its cube, etc. $\overset{\overset{1}{\smile}}{6}\ p\ 4$ meant $6x + 4$; $\overset{\overset{3}{\smile}}{12}\ p\ \overset{\overset{2}{\smile}}{22}\ m\ 4$ meant $12x^3 + 22x^2 - 4$. He used p and m, the initial letters of plus and minus, in place of our usual signs.

The illustrations on page 50, taken from Bombelli's Algebra printed in 1579, show two problems in addition and one in subtraction.

Stevinus, a few years later, used numbers in circles to denote the powers. He wrote $3x^2 + 4x - 5$ as $3\ \textcircled{2} + 4\ \textcircled{1} - 5\ \textcircled{0}$.

III. Vieta

In his History of Mathematics, Ball speaks of how much Vieta, a Frenchman, improved algebraic symbols, when he let A represent an unknown number, and A *quadratus* and A *cubus* represent its square and cube, respectively. Ball says, "Vieta would have written the equation

$$3BA^2 - DA + A^3 = Z,$$

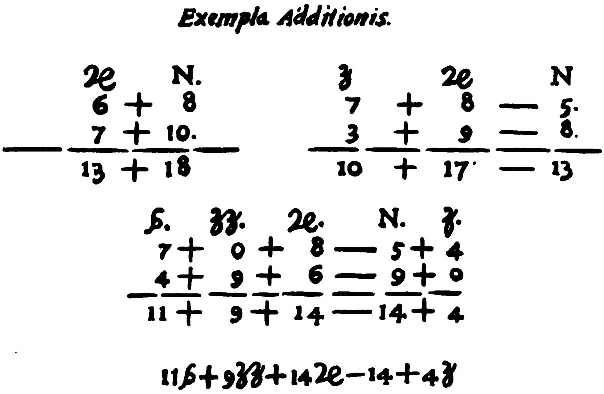
as *B3 in A quad. − D plano in A + A cubo aequatur Z solido.*"

IV. Clavius

Vieta's symbols do seem an improvement when we compare them with the special characters used for each different power by Christophorus Clavius in his book on mathematics published in 1611.

The following illustrations taken from his book give problems in addition and in multiplication.

ADDITION BY CLAVIUS



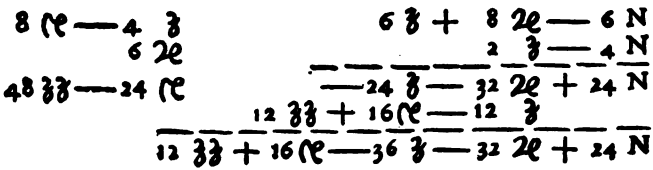
In modern symbols these problems are as follows:

$$\begin{array}{r} 6x + 8 \\ 7x + 10 \\ \hline 13x + 18 \end{array} \qquad \begin{array}{r} 7x^2 + 8x - 5 \\ 3x^2 + 9x - 8 \\ \hline 10x^2 + 17x - 13 \end{array}$$

$$\begin{array}{r} 7x^5 + 0x^4 + 8x - 5 + 4x^2 \\ 4x^5 + 9x^4 + 6x - 9 + 0x^2 \\ \hline 11x^5 + 9x^4 + 14x - 14 + 4x^2 \end{array}$$

The form of multiplication used by Clavius is still more complex, as shown by the following illustration.

MULTIPLICATION BY CLAVIUS



In our algebraic language these problems would be written thus:

$$\begin{array}{r} 8x^3 - 4x^2 \\ 6x \\ \hline 48x^4 - 24x^3 \end{array} \qquad \begin{array}{r} 6x^2 + 8x - 6 \\ 2x^2 - 4 \\ \hline -24x^2 - 32x + 24 \\ 12x^4 + 16x^3 - 12x^2 \qquad\qquad \\ \hline 12x^4 + 16x^3 - 36x^2 - 32x + 24 \end{array}$$

When we think of the difficulties the mathematicians of the earlier times had on account of their lack of suitable symbols, we marvel at the extent of their knowledge and appreciate modern algebraic language more.

CHAPTER THREE

THE EQUATION

A. HOW TO SOLVE AN EQUATION

1. (a) Six times a number decreased by two equals four times the number increased by 8.

(b) *Solution:*

Let $n =$ the number.
Then, $6n - 2 = 4n + 8.$

To solve this equation, we must add or subtract whatever is necessary to make a new but true equation that will have only literal terms in one member, preferably the first, and only numerical terms in the other. In other words, we must add $+ 2$ to both members to offset the $- 2$ in the first member; and we must subtract $4n$, or add $- 4n$, to both members to offset the $+ 4n$ in the second member.

$$\begin{array}{rcll} 6n - 2 &=& 4n + 8 & \\ +2 &=& +2 & \\ \hline 6n &=& 4n + 10 & \qquad\text{Add. Ax.} \\ -4n &=& -4n & \\ \hline 2n &=& 10 & \qquad\text{Add. Ax.} \\ \dfrac{\cancel{2}n}{\cancel{2}} &=& \dfrac{\overset{5}{\cancel{10}}}{\cancel{2}} & \qquad\text{Div. Ax.} \\ n &=& 5 & \end{array}$$

(c) If an equation had very many terms which were in the wrong members, solving the equation would

become a tedious task. Let us see if we can find a shorter method.

If we merely indicate the addition of 2 to each member, the equation is

$$6n - 2 + 2 = 4n + 8 + 2$$

Since $\qquad -2 + 2 = 0$

we may drop these numbers, and have

$$6n = 4n + 8 + 2.$$

Compare this equation with the original one,

$$6n - 2 = 4n + 8.$$

We see that the -2 that is dropped from the first member appears in the second member as $+2$.

In the new equation,

$$6n = 4n + 10,$$

we need to add $-4n$ to both members; as,

$$6n - 4n = 4n - 4n + 10.$$

Since the two terms, $+4n - 4n$, equal 0, we may drop them, also.

In the new equation,

$$6n - 4n = 10,$$

the $+4n$ dropped from the second member appears in the first member as $-4n$, that is, with its sign changed.

(d) May any term be dropped from one member of an equation, if the same term with its sign changed is placed in the other member? Why?

(e) Such a transfer of terms from one member of an equation to the other by changing their signs is called *transposition*. *Transposition* comes from two Latin words which mean *placing across*, that is, across the bridge of the equality sign. Any

term may cross the bridge provided it pays the toll of changing its sign.

(f) All like terms in each member of an equation should be combined before any term is transposed.

(g) It should never be forgotten that transposition is simply a short way of applying the addition and subtraction axioms.

(h) To solve an equation is to find the value of the unknown or literal term or terms.

I. Problems

Solve the following equations and check the result in each.

1. Seven times a number increased by 5 is 33. Find the number.

2. A number decreased by 25 equals 54. Find the number.

3. Nine times a number decreased by the number is equal to six times the number plus 12. What is the number?

4. 48 exceeds 3 times a number by as much as the number exceeds 12. What is the number?

Solution:

Let	$n = \text{the number}$
Then,	$48 - 3n = n - 12$
Transposing,	$-3n - n = -12 - 48$
Combining like terms,	$-4n = -60$
Multiplying by -1,	$4n = 60$
Dividing by 4,	$n = 15$

Since both members of an equation may be multiplied by -1, it means that the signs of all the terms in an equation may be changed.

5. 52 exceeds 5 times a number by as much as 34 exceeds 3 times the number. What is the number?

6. Five times a number decreased by 2 exceeds three times the number by as much as 46 exceeds the number. Find the number.

7. The sum of two numbers is 28. The larger is 2 more than the smaller. What are the numbers?

8. A farmer raised twice as many bushels of corn as of oats; and 3 times as many bushels of wheat as of corn. If he raised 981 bushels of grain, how many bushels of each kind did he raise?

9. A yard stick is broken into two parts, one of which is 10 in. longer than the other. What is the length of each part?

10. A boy is 6 years older than his brother. The sum of their ages is 28 years. How old is each?

11. The sum of three consecutive numbers is 240. What are the numbers?

12. The sum of three consecutive even numbers is 276. What are the numbers?

13. The sum of four consecutive odd numbers is 208. What are the numbers?

14. (a) Four-fifths of a number plus 7 is 71. What is the number?

(b) *Solution:*

$$
\begin{array}{lrcll}
\text{Let} & n & = & \text{the number.} & \\
\text{Then} & \frac{4}{5}n + 7 & = & 71 & \\
 & \cancel{5}\cdot\frac{4}{\cancel{5}}n + 5\cdot 7 & = & 5\cdot 71 & \text{Mult. Ax.} \\
 & 4n + 35 & = & 355 & \\
 & 4n & = & 355 - 35 & \text{Subt. Ax.} \\
 & 4n & = & 320 & \\
 & n & = & 80 & \text{Div. Ax.} \\
\textit{Check.} & \frac{4}{5}\cdot 80 + 7 & = & 71 & \\
 & 64 + 7 & = & 71 & \\
 & 71 & = & 71 &
\end{array}
$$

(c) Why is it necessary to multiply each term of the equation by 5?

(d) Explain how each equation was derived from the one preceding.

(e) Try to solve the equation by first transposing the 7 and combining it with the 71.

(f) Which method is shorter?

(g) Does it pay to examine an equation before solving it to find the best method?

15. (a) Five-eighths of a number decreased by 5 equals $\frac{1}{2}$ the number plus 4.

$$\tfrac{5}{8}x - 5 = \tfrac{1}{2}x + 4.$$

(b) By what number may you multiply each term to get rid of the fractions, that is, to clear the equation of fractions?

(c) Clear the equation of fractions first and solve completely.

(d) What other steps might be taken before clearing the equation of fractions? Solve in this way and compare with the first method.

(e) Check the equation.

16. Five-sixths of a number decreased by 5 equals $\frac{3}{4}$ of the number plus 3. What is the number?

17. A number plus $\frac{3}{8}$ of itself decreased by 8 equals $\frac{3}{4}$ of the number plus 12. Find the number.

18. A coat sold for $60, which was at a gain of 20 % above cost. What was the cost?

Suggestion: If c = the cost, what fractional part of c would equal the gain?

Write a formula which gives a general rule for finding the selling price from the cost and the gain.

19. Think of some number. Multiply it by 7. Add 8. Subtract 5 times the number. Subtract 4. Add the number. Add 6. Subtract 3 times the number. Add 2 and divide by 2. Your result is 6. Show why the result is 6, regardless of the number you started with. Try this puzzle on some friend.

20. Make up similar puzzles.

21. Two boys went fishing and caught 23 fish. One boy caught three times as many as the second and 3 more. How many did each catch?

22. Two girls working during vacation earn $115. One girl earns $5 less than twice as much as the other girl. How much does each earn?

23. Two boys grading a lot back of a new home hauled away 45 wheelbarrows of dirt. One boy hauled 3 wheelbarrows less than twice as many as his friend. How many did each haul?

24. Two baseball teams won 50 games. One won twice as many as the other and two more. How many did each win?

25. Three boys, A, B, and C solved 40 problems. A solved twice as many as B, and C solved one third as many as B. How many did each solve?

26. John and James solved 55 problems in an hour. If John solved 10 more than twice as many as James, how many did each solve?

27. A dress and a hat cost $20. If the dress cost $1 less than three times as much as the hat, what was the cost of each?

28. A boy pays $4.50 for a baseball and glove. Five-sixths of the cost of the glove plus 50¢ equals two times the cost of the ball. What is the cost of each?

29. A girl spends $15 for a hat and pair of shoes. The shoes cost one and one half times as much as the hat. Find the cost of each.

30. A boy's suit was sold for $20 after a discount of 20% was taken from the market price. What was the market price?

31. A furniture store advertised during the month of August that it would give discounts on all articles sold for cash. Find the marked price if the following sale prices and discounts are given.

	Article	Sale Price	Discount	Marked Price
(a)	Kitchen cabinet	$31.50	10 %	....
(b)	Kitchen table	12.15	10 %	....
(c)	Writing desk	40.00	20 %	....
(d)	Magazine rack	8.00	20 %	....
(e)	Davenport	50.00	$33\frac{1}{3}$ %	....
(f)	Table lamp	8.82	30 %	....
(g)	Gas range	29.85	25 %	....
(h)	Brussels rug	25.00	$33\frac{1}{3}$ %	....
(i)	Rocker	19.80	10 %	....
(j)	Dresser	36.00	20 %	....

32. How long will it take $500 to gain $200 at 5 % simple interest?

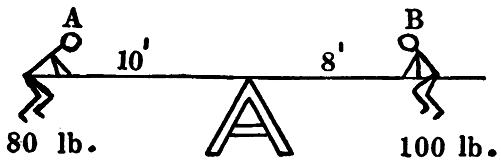
33. A man has his money invested in a building which brings in $4000 a year in rent. His expenses in insurance, taxes, and general repairs are $1000 a year. He has an opportunity to sell the property and invest the money at 6 % interest. How much must he realize from the sale to make the annual income from both investments the same?

34. (a) What is a teeter board? To make it balance, which part of the board is placed over the support?

(b) If the middle of the board is supported, and two boys of unequal weights sit at the ends of the board, will it be in balance?

(c) How can the lighter-weight boy balance the heavier one?

(d) A teeter board 20 ft. long is supported in the middle. At one end of the board is a boy who



weighs 80 lb. A boy weighing 100 lb. finds he must move 2 ft. from the other end, in order to balance the lighter boy.

(e) Multiply the weight of each boy by his distance from the support. How do the two products compare?

(f) Any bar or rod resting on a support, as a teeter board, is called a *lever*. The point of support is a *fulcrum*.

(g) A teeter board is supported in the middle. A boy weighing 96 lb. is 10 ft. from the fulcrum. How far from the fulcrum must the boy on the other end be, if he weighs 80 lb.?

Solution: Given A's weight = 96 lb.
A's distance = 10 ft.
B's weight = 80 lb.

To find B's distance from the fulcrum.

Let x = no. of ft. in B's distance.

Then, $80 \cdot x = 96 \cdot 10$ Why?

$$x = \frac{\overset{12}{\cancel{96}} \cdot \cancel{10}}{\underset{8}{\cancel{80}}}$$

$\therefore\ x = 12$ ft.

(h) To balance a boy of 96 lb. at 10 ft. from the fulcrum, how heavy must the second boy be if he sits 12 ft. from the fulcrum?

Solution: Given A's weight = 96 lb.
A's distance = 10 ft.
B's distance = 12 ft.

To find B's weight.

Let n = no. of lb. in B's weight.

Then, $12n = 96 \cdot 10$

$$n = \frac{\overset{8}{\cancel{96}} \cdot 10}{\cancel{12}}$$

$\therefore\ n = 80$ lb.

(i) In the following problems find the missing weight or distance.

A's Weight	B's Weight	A's Distance from Fulcrum	B's Distance
90 lb.	120 lb.	12 ft.	.....
100 lb.	......	14 ft.	10 ft.
......	90 lb.	18 ft.	17 ft.
75 lb.	105 lb.	.....	5 ft.
112 lb.	144 lb.	9 ft.	.....

II. Practice in Equations

Thus far you have used those equations which have only the first power of the unknown or literal terms. An equation which contains unknown or literal terms which are of the first degree only is a *simple equation.*

A simple equation may be called an equation of the *first degree*. Why?

When you solve an equation you find a value of the unknown term. If this value checks, it is called a *root* of the equation. When the value of an unknown checks it is said to satisfy the equation.

The *formula* is another name for the equations used by mechanics, artisans, and engineers. The equation is used to solve many problems that are impossible with arithmetic methods only. You will find that it is used very extensively in all branches of mathematics. It is the very foundation of mathematics. Since it is so essential to have skill in handling equations, the following problems are given for practice. Solve without checking. Note the time taken.

1. $3x - 6 = x + 4$
2. $5n + 3 = 2n + 15$
3. $8a - 25 = 3a - 5$
4. $11c + 4 = 5c + 9$
5. $\frac{7n}{2} + 6 = 4n - 5$

6. $4a - 6 + a = 3a + 5 - 8a + 4$
7. $5x + 8 = 9x - 10$
8. $7b - 3 = 15 + 10b$
9. $8x + 6 = -12 + 4x$
10. $6n + 8 = 3n + 20$
11. $15a - 6 = 12a + 12$
12. $\frac{3x}{2} - 8 = 4x - 9$
13. $\frac{5x}{3} - 2 = \frac{3x}{2} + 4$
14. $3x - 2 + x = 6 + x - 10$
15. $9n + 7 + 2n = 8 + 6n + 14$
16. $6a + \frac{1}{2} = 5a + \frac{2}{3}$
17. $12b - 6 = 4b + 10$
18. $5x - 7 = 10x + 8$
19. $19c + 11 = 14c - 14$
20. $\frac{8x}{3} + 6 = \frac{2x}{3} - 9$

21. Check the foregoing equations.
22. (a) Solve equations 1 to 20 inclusive on alternate days for two weeks, noting the exact number of minutes taken each time.
(b) Make a graph of your speed record.

III. Achievement Test

After having sufficient practice on these equations, test yourself on the following list or use Rugg's Standard Test No. 2.

Twelve or more solved correctly in 5 minutes show satisfactory achievement. Pupils solving less than 12 should have further practice.

1. $4x + 7 = 2x + 11$
2. $8a - 3 = 4a + 9$

3. $6b - 30 = 2b - 2$
4. $12n + 8 = 7n + 18$
5. $\frac{3x}{8} - 2 = \frac{1}{2} - \frac{5x}{8}$
6. $7a + 3 - 2a = 3 + 2a - 7 + 2$
7. $9n + 6 = 5n - 15$
8. $10x + 6 = 7x + 12$
9. $18c - 7 = 12c + 8$
10. $7x - 4 + 2x = 8 + 2x - 17$
11. $5b - \frac{1}{3} = 10b + \frac{1}{2}$
12. $17a + 12 = 15a - 20$
13. $8n - 6 = 10n - 8$
14. $4x + 8 + 2x = 3x - 10$
15. $10t - 9 = 15t - 6t$
16. $\frac{x}{6} - \frac{2}{3} = 4x + 7$
17. $42y + 16 - 2y = 28y + 40$
18. $\frac{7a}{8} - 6 = 2a - 15$

B. GRAPHS OF SIMPLE EQUATIONS

I. Vertical and Horizontal Number Scales

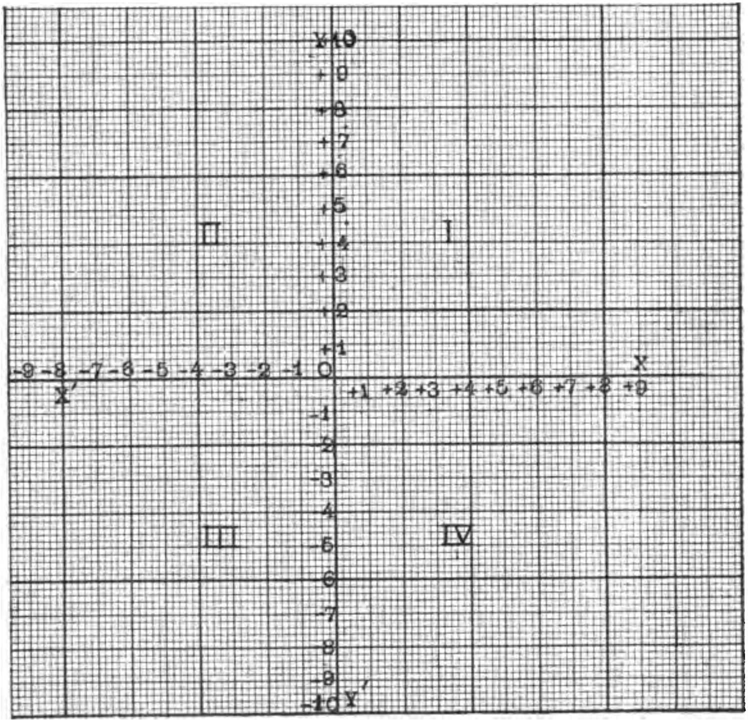
1. You have made pictures or graphs of changes in temperature and of records of various kinds. In the same way, it is possible to make graphs of equations. You have counted positive and negative numbers to the right and left respectively on the horizontal number scale. You have counted them up and down respectively on the thermometer or vertical number scale.

Make a picture of both of these scales crossing each other at zero.

2. These two number scales divide the space around zero into four quarters of *quadrants*. The upper right hand quadrant is the first. The quadrants are numbered I, II,

III, and IV, beginning at the first and counting in a counter-clockwise direction, that is, in the direction opposite to that of the hands of a clock.

Until you learned about negative numbers all of your measurements and calculations were in the first quadrant.

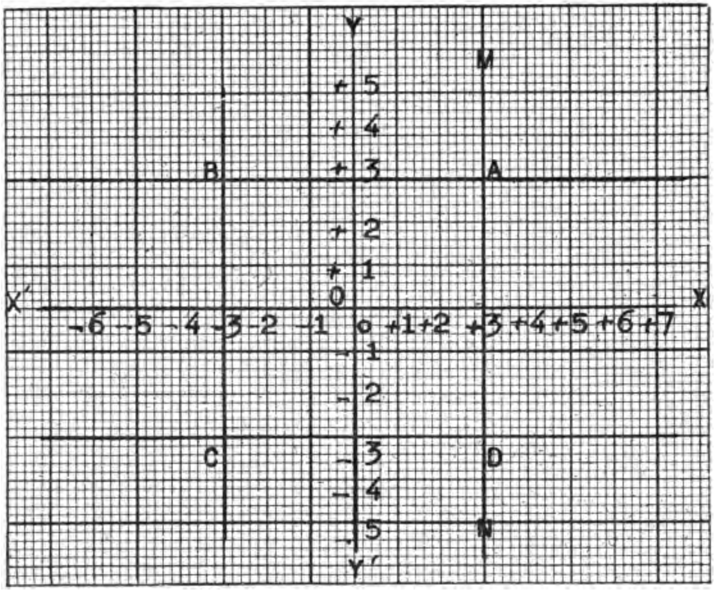


Since you had no need for the three other quadrants, it was unnecessary to draw even the first.

For convenience the horizontal scale is lettered XX′ and is called the *x-axis*; the vertical scale is lettered YY′ and is called the *y-axis*.

II. Location of Points

1. (a) On squared paper draw a set of axes and number the spaces in the four directions from zero or point O. Point O is the beginning point for all counting. It is therefore called the *origin.*
   (b) If x is $+3$, x is located by counting 3 to the right of O, or zero. Through this point, $+3$ on the x-axis, draw a line, MN, parallel to the y-axis. Any point in this line is 3 spaces to the right of the y-axis; that is, any point in this line represents $x = +3$. The line contains all points whose distance from the y-axis is 3 units to the right; therefore the line MN is the picture or graph of the equation $x = +3$.
   (c) An equation must be in its simplest form before its graph is made.
2. (a) If x is -3, how would you count to locate it?
   (b) Draw a graph of the equation $x = -3$.
   (c) What direction from the y-axis is it? How far from it?
3. (a) If $y = +3$, y is located by counting 3 above O or zero.
   (b) Through this point, $+3$ on the y-axis, draw a line parallel to the x-axis.
   (c) Why is this line the graph of the equation $y = 3$?
4. (a) If y is -3, how would you count to locate it?
   (b) Draw a graph of the equation $y = -3$.
5. (a) The four graphs intersect and form a square *ABCD.* See figure.
   (b) In which quadrant is the corner A located? Locate B, C, and D in their respective quadrants.
   (c) Each of these points may be exactly located if we know its distance and direction from each axis. For point A, the distance measured on the x-axis is $+3$, and the distance measured on the y-axis



is + 3. For brevity, we speak of these distances as the *x-distance* and *y-distance.*

(d) The sign of the 3 shows the direction.

(e) For point *B*, the *x*-distance is − 3 and the *y*-distance is + 3.

(f) What is the *x*-distance for point *C*? Give its *y*-distance.

(g) Give each distance for point *D*.

6. In tabular form these distances are

Point	x-distance	y-distance
A	3	3
B	− 3	3
C	− 3	− 3
D	3	− 3

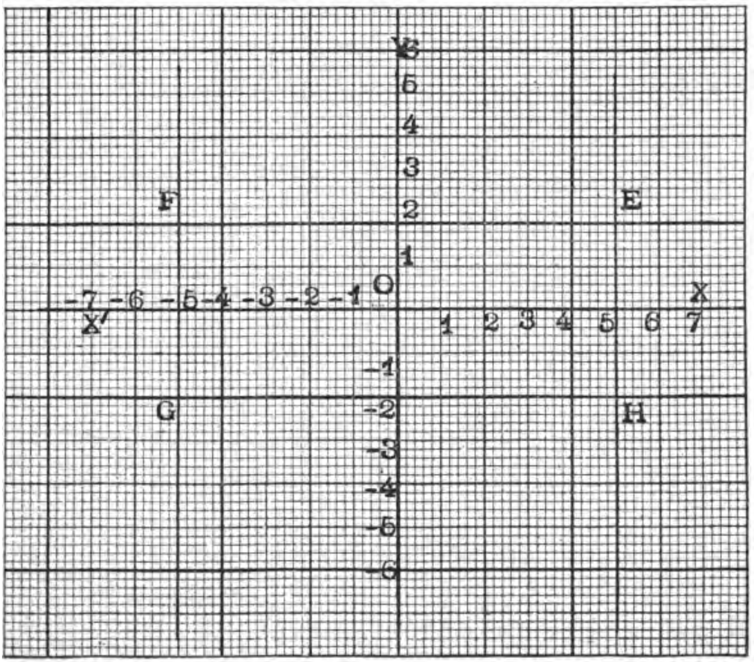
7. The location of a point may be given more briefly by writing these two distances in parentheses thus:

$$A = (3, 3)$$
$$B = (-3, 3)$$
$$C = (-3, -3)$$
$$D = (3, -3)$$

In this form the x-distance is always given first.

8. (a) Draw the graphs of the following equations on the same axes:

(1) $x = 5$
(2) $x = -5$
(3) $y = 2$
(4) $y = -2$



(b) Letter the intersections of these graphs E, F, G, and H.

(c) What shaped figure is inclosed?

(d) The locations of these points are:

$$E = (5, 2)$$
$$F = (-5, 2)$$
$$G = (-5, -2)$$
$$H = (5, -2)$$

9. (a) Draw graphs of each of the following sets of equations, using one pair of axes for a set.

(b) Give the locations of the points of intersection in each.

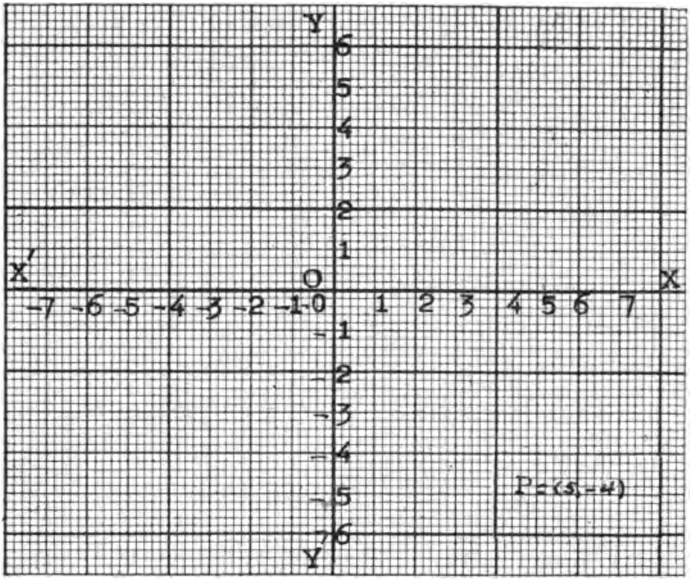
(1) $x = 4, x = -4, y = 5, y = -5$
(2) $x = 6, x = -6, y = 6, y = -6$
(3) $x = 8, x = -8, y = 7, y = -7$
(4) $x = 5, x = -5, y = 3, y = -3$
(5) $x = 7, x = -7, y = 7, y = -7$
(6) $x = 2, x = -2, y = 5, y = -5$
(7) $x = 3, x = -3, y = 4, y = -4$
(8) $x = 9, x = -9, y = 6, y = -6$
(9) $x = 7, x = -7, y = 4, y = -4$
(10) $x = 6, x = -6, y = 3, y = -3$

10. (a) Draw a pair of axes.

(b) Locate a point P if $P = (5, -4)$. The 5 tells us to count 5 places to the *right* on the x-axis. From this point count *down* 4 places on a line parallel to the y-axis, as the (-4) directs. Place a dot at this point, P, whose position is $(5, -4)$. (See graph on page 71.)

11. On one pair of axes locate the following points.

$A = (3, 4)$	$D = (-2, -3)$
$B = (-5, 6)$	$E = (6, 8)$
$C = (4, -7)$	$F = (-4, -4)$



$G = (0, 4)$	$L = (7, -3)$
$H = (-4, 2)$	$M = (-5, -8)$
$I = (-2, 4)$	$N = (4, 9)$
$J = (-3, 0)$	$O = (0, 0)$
$K = (-2, -5)$	$P = (0, -5)$

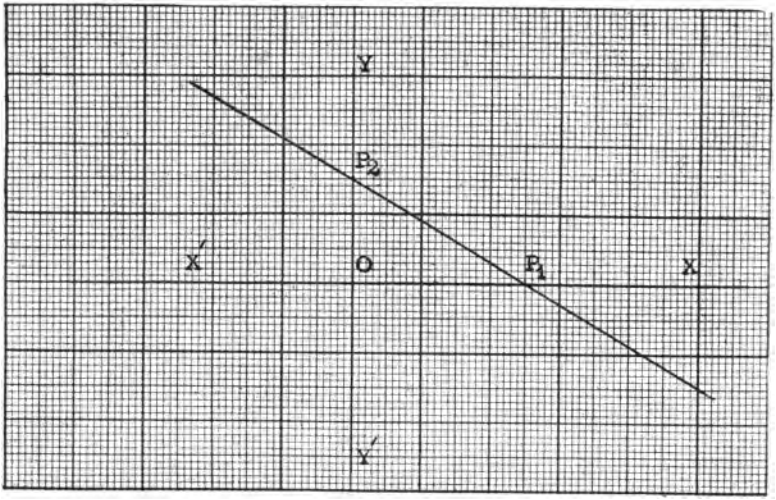
12. (a) Draw a pair of axes and locate two points as follows:

$P_1 = (5, 0)$ and $P_2 = (0, 3)$

(See graph on page 72.)

(b) Draw a line between the two points.

13. (a) Draw axes and locate the following pairs of points.



(b) Draw a line through each pair of points.

No.	P_1	P_2
(1)	(– 3, 0)	(0, 6)
(2)	(– 4, 0)	(0, – 5)
(3)	(7, 0)	(0, – 3)
(4)	(0, 0)	(6, 4)
(5)	(0, 0)	(– 5, 3)
(6)	(0, 0)	(5, – 3)
(7)	(0, 0)	(– 2, – 6)
(8)	(6, 2)	(– 3, 4)
(9)	(– 6, 2)	(3, – 4)
(10)	(5, 3)	(– 3, – 5)

C. MEANING OF THE SIMPLE EQUATION

I. Equations with One Unknown

1. You have seen that every simple equation having only one unknown quantity may be simplified to the form of $x = a$, in which a means a known number. You have found

that the graph of such a simple equation is a straight line. Because its graph is always a straight line, the simple equation is called a *linear* equation. Why is linear an appropriate name for it?

2. What is a third name for a simple equation? What does this last name tell about the equation?

II. Equations with Two Unknowns

1. A simple equation may have more than one unknown quantity, but no term of the equation may have more than one unknown factor. $x + y = 12$ and $3x + 4y = 32$ are linear equations.

2. A number of different stories might be written about the equation $x + y = 12$. It might mean that the sum of the length and width of a rectangle is 12 inches; or that a boy had earned x dollars from selling beans and y dollars from selling tomatoes from his garden, and from both he received 12 dollars; or that the sum of two numbers is 12.

3. Make up original stories about this equation.

4. No matter what story the equation, $x + y = 12$, represents, its geometrical picture is always the same, and, because it is a simple equation, the picture is a straight line.

5. (a) Since there is only one equation about these two unknowns, we cannot know the values of x and y, but we do know that their sum is 12, and as the value of one changes, the value of the other changes also. If x is 1, then y is 11. If x is 4, then y is 8.

(b) Any value from 0 to 12, fractional or integral may be assumed for x, and for each changing value for x, the value of y will change or vary also.

(c) The following table shows the possible positive integral values for x and y.

$$x + y = 12$$

x	y
0	12
1	11
2	10
3	9
4	8
5	7
6	6
7	5
8	4
9	3
10	2
11	1
12	0

(d) Besides these values, there are innumerable fractional and negative values that might be assumed; so the possible roots of this equation are infinite in number.

(e) Because the value of one term varies with a change of value of the other, the two terms x and y are called *variables*. An equation with two unknown terms is called an *equation of two variables*.

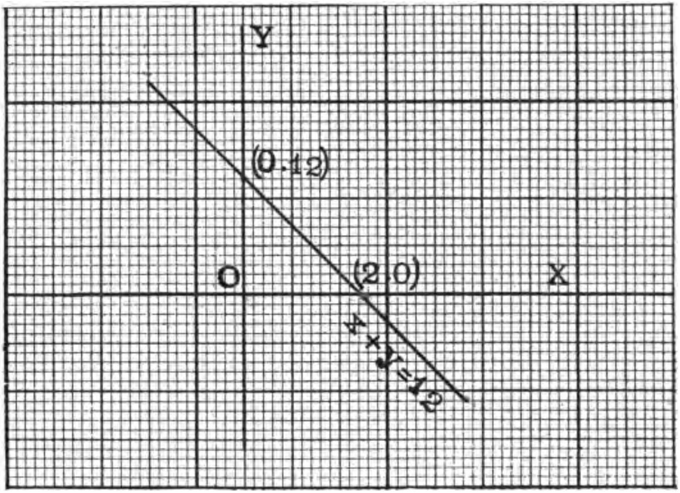
III. Graph of a Simple Equation of Two Variables

1. (a) To make a graph of $x + y = 12$, on a set of axes plot any two points given in the foregoing table.

(b) Why are two points sufficient?

(c) Any two points will fix the position of the line, but the drawing will be more accurate if the two points are not too close together.

(d) The two points (0, 12) and (12, 0) are the farthest apart and are conveniently plotted. Through these two points draw a straight line. This line is the graph of the equation.



GRAPH OF $x + y = 12$

(e) From this graph read the value of y when x is 5, 2, 7, 8. Read the value of x when y is 10, 4, 6, 3.

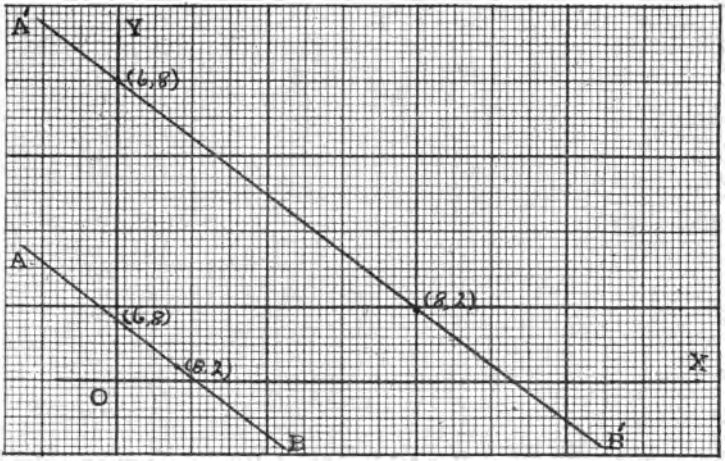
2. (a) Graph the equation $3x + 4y = 32$.

$3x + 4y = 32$

x	y
0	8
...	...
...	...
...	...
...	...
...	...
8	2
$9\frac{1}{3}$	1
$10\frac{2}{3}$	0

(b) Since, when $x = 0$, $y = 8$, we may use (0, 8) for one point and need not assume more values for x. The substitution of $y = 0$ gives $x = 10\frac{2}{3}$. Since

this value is fractional, it is a little more difficult to plot the point. Substitute in order $y = 1$ and $y = 2$, until you find a value for y that gives an integral value for x. Plot the points (0, 8) and (8, 2) and draw the line of the equation. See AB in the figure.



GRAPH OF $3x + 4y = 32$

(c) Because this graph is small, it is difficult to read exact values from it. Instead of counting each small space as a unit, if we count each of the larger spaces as one, we get the line $A'B'$ as the graph of the equation.

(d) How do these two lines compare in position?

(e) Use the small spaces as units, and on this same pair of axes, graph the equation $15x + 20y = 160$. How does this line compare with the other two in position?

(f) The fact that these lines take the same direction, or are parallel, shows that the equations are

practically the same, that is there is no change in the relation of the two variables x and y. Notice that $15x + 20y = 160$ is the first equation multiplied by 5. These pictures show graphically that in multiplying both members of an equation by the same number, we do not change the equation.

(g) Show that they prove also that both members of an equation may be divided by the same number.

3. Draw graphs of the following equations:

(a) $x + y = 14$	(f) $x + 5y = 11$	(k) $x + 2y = 8$
(b) $2x + 7y = 20$	(g) $6x + y = 16$	(l) $3x + 2y = 18$
(c) $4x + 14y = 40$	(h) $3x + 2y = 12$	(m) $x + 5y = 13$
(d) $5x + 6y = 50$	(i) $6x + 7y = 42$	(n) $8x + 9y = 42$
(e) $5x + 6y = 18$	(j) $10x + 3y = 30$	(o) $7x + 5y = 66$

D. QUADRATIC EQUATIONS

I. Derived from Problems

Some problems give equations of a kind other than simple or linear. Take, for example, this problem: If x is the side of a square, what is its area? If the area is 25 square inches, how long is each side?

Since this equation is about x^2, it cannot be an equation of the first degree, that is, a linear equation.

Solution:

Let x = no. of inches in side of □.

Then, x = no. of sq. in. in area of □.

Given 25 = no. of sq. in. in area of □.

$\therefore x^2 = 25.$

To find the value of x, it is necessary to take the square root of both sides of the equation.

$$\therefore x = +5 \text{ or } -5$$

Proof.

$$(+5)^2 = 25$$
$$(-5)^2 = 25$$

II. Positive and Negative Square Roots

1. Draw a pair of axes. In the first quadrant draw a square on the line which is $+5$ units long. What is the area of this square?

2. On the x-axis, count -5 units. On the y-axis, count -5 units. In the third quadrant make a square with these two lines as sides. What is the area of this square?

3. In the first quadrant,

$$x = +5 \text{ and } x^2 = 25.$$

In the third quadrant,

$$x = -5 \text{ and } x^2 = 25.$$

We see, therefore, that any positive number may have two square roots, one positive and the other negative.

4. Instead of writing $\sqrt{25} = +5$ or -5, we may write the root once and put the double sign in front of it, thus: $\sqrt{25} = \pm 5$. This statement is read *the square root of* 25 *is* (*or equals*) *plus* or *minus* 5.

III. Kinds of Equations of the Second Degree

The problem about the square gives us the equation, $x^2 = 25$, which is not simple. It is called a *quadratic equation*, because it contains the second degree, or the square, of the unknown term.

1. A *quadratic equation* must contain a *second degree term* but no higher power. If it contains *only* the second power, as $x^2 = 16$, it is an *incomplete quadratic equation.*

2. If it contains both the first and second powers, as $x^2 + 3x = 180$, it is a *complete quadratic.*

3. Later we shall see that the graph of a quadratic equation is not a straight line, but is some kind of curved line.

IV. Problems in Incomplete Quadratic Equations

1. If the area of a circle is 616 square inches, what is its radius?

(a) *Solution:*

$$S_{\bigcirc} = \pi r^2$$

$$\text{Given } S_{\bigcirc} = 616$$

$$\therefore \pi r^2 = 616$$

$$\frac{22}{7} r^2 = 616$$

Multiplying both sides by $\frac{7}{22}$,

$$\frac{7}{\cancel{22}} \times \frac{\cancel{22}}{7} r^2 = \overset{28}{\cancel{616}} \times \frac{7}{\cancel{22}}$$

$$r^2 = 196$$

$$\sqrt{r^2} = \sqrt{196}$$

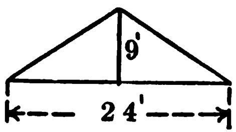
$$r = \pm 14$$

$$\therefore \text{radius} = 14 \text{ inches}$$

(b) Why should the negative root be discarded?

2. Find the hypotenuse of a right triangle whose base is 40 ft. and altitude is 30 ft.

3. The surface of a sphere is 2464 sq. ft. Find the radius, using the formula for the surface, $S = 4 \pi r^2$.



4. The gable end of a building is 24 ft. Its height is 9 ft. Find the length of the hip rafters.

5. How much fencing is needed to inclose a 40-acre farm, which is in the shape of a square?

6. A cistern can be made 9 ft. deep. What radius must it have to enable it to contain $2828\frac{4}{7}$ cu. ft. of water? (The vol. of a cyl. $= \pi r^2 h$.)

7. Sand is piled in the shape of a cone till it is 9 ft. high. The volume of this pile is 462 cu. ft. How large a radius will the base of the sand pile require? ($V_{co} = \frac{1}{3} \pi r^2 h$.)

8. A balloon is 1156 ft. high. How long will it take a stone dropped from it to reach the ground? The force of

gravity is approximately 32. Use the formula $s = \frac{1}{2}gt^2$. (s = distance in feet and t = time in seconds.)

9. A boy wants a rabbit pen to contain 80 sq. ft. and have a length of 10 ft. How wide must it be? If he wishes to make this into two pens by putting wire diagonally across it, how many feet of wire are needed?

10. A well is 21 ft. deep and contains 594 cu. ft. of water. What radius does it have?

11. With a 45° right triangle, find the height of your school building. How long will it take a ball to fall from the top of the building to the ground?

V. Practice Exercises in Incomplete Quadratics

1. $x^2 = 225$	**11.** $9v^2 - 2 = 8v^2 + 2$
2. $3x^2 = 588$	**12.** $15a^2 + 5 = 9a^2 + 35$
3. $a^2 - 16 = 384$	**13.** $\frac{22r^2}{7} = 201\frac{1}{7}$
4. $5b^2 + 20 = 2900$	**14.** $x^2 = 2874$
5. $8y^2 + 16 = 2y^2 + 46$	**15.** $3a^2 = 896$
6. $c^2 = 256$	**16.** $9y^2 + 1 = 4y^2 + 56$
7. $\frac{22r^2}{7} = 2464$	**17.** $\frac{22r^2}{7} = 345$
8. $16t^2 = 3600$	**18.** $16t^2 = 8100$
9. $7x^2 - 67 = 4x^2 + 176$	**19.** $8t^2 - 4 = 6t^2 + 10$
10. $8y^2 - 42 = 38$	**20.** $4x^2 + 9 = 2x^2 + 13$

VI. Complete Quadratic Equations

1. In the first discussion of the quadratic equation, $x^2 + 3x = 180$ was given as an illustration of a complete quadratic.

Let us see what kind of a problem might give this equation.

(a) If a rectangle is 3 ft. longer than wide, how may you represent the two dimensions?

(b) What area do these dimensions give?
(c) If the area is 180 sq. ft., what are the dimensions?
(d) *Solution:*

$$\begin{array}{lrl}
\text{Let} & x & = \text{no. of ft. in width.} \\
\text{Then,} & x + 3 & = \text{no. of ft. in length.} \\
 & x(x + 3) & = \text{no. of sq. ft. in area.} \\
\text{Given,} & 180 & = \text{no. of sq. ft. in area.} \\
\therefore & x(x + 3) & = 180 \\
 & x^2 + 3x & = 180
\end{array}$$

If the 180 is transposed to the first member, it will make a quadratic trinomial, such as you have learned to factor.

$$x^2 + 3x - 180 = 0.$$

Factor the first member, thus:

$$(x + 15)(x - 12) = 0.$$

If the product of two factors is 0, what must one of the factors be?

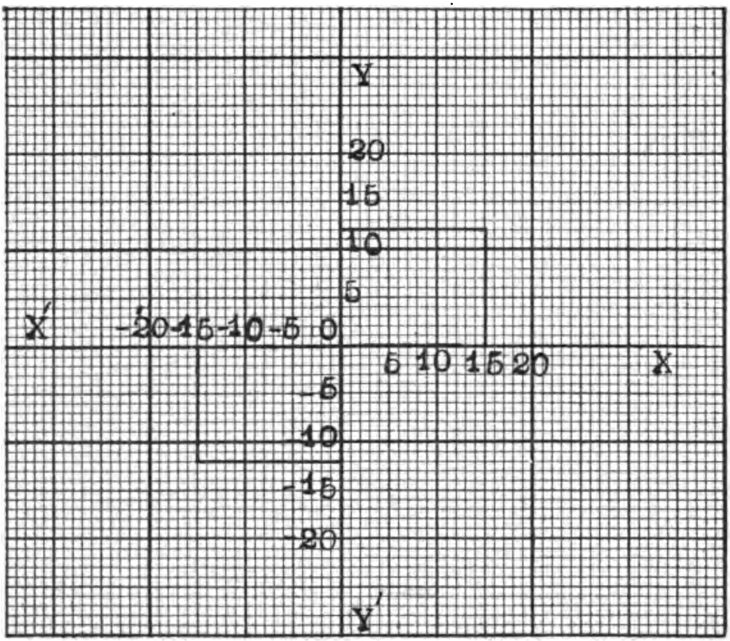
If $(x + 15)(x - 12) = 0$, either $x + 15$ must equal 0, or $x - 12$ must be 0; otherwise the product could not be zero. Unless one of two numbers is zero, their product cannot equal zero.

$$\begin{array}{lrl}
\text{If} & x + 15 & = 0 \\
\quad\text{then} & x & = -15 \\
\quad\text{and} & x + 3 & = -15 + 3 \\
 & & = -12 \\
\quad\text{and} & x(x + 3) & = (-15)(-12) \\
\text{If} & x - 12 & = 0 \\
\quad\text{then} & x & = 12 \\
\quad\text{and} & x + 3 & = 12 + 3 \\
 & & = 15 \\
\quad\text{and} & x(x + 3) & = (12)(15).
\end{array}$$

2. By using factoring to solve this quadratic, we get two possible rectangles:

$$
\begin{aligned}
S_{\square} &= lw \\
S_{\square} &= (-15)(-12) \\
&= 180 \\
S_{\square} &= (15)(12) \\
&= 180
\end{aligned}
$$

(a) On a pair of axes draw a rectangle in the third quadrant whose dimensions are -15 and -12. In the first quadrant, draw one whose dimensions are 15 and 12.



(b) In a simple equation, how many values for the unknown can be found? How many values can be found for the unknown in a quadratic equation?

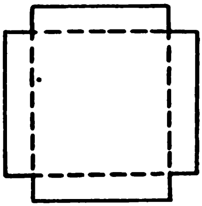
(c) The solution of any quadratic equation gives two roots, but if the thought in the problem makes the negative root absurd or meaningless, it is dropped. In all other cases both roots are used.

VII. Problems in Complete Quadratics

1. If a rectangle is 5 ft. longer than wide, and the area is 84 sq. ft., what are the two dimensions?

2. A rectangle is 8 ft. longer than wide. Its area is 240 sq. ft. What are the dimensions?

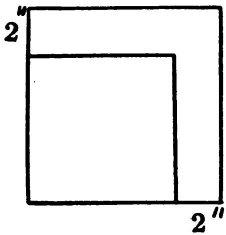
3. The area of a triangular field is $1\frac{1}{2}$ acres. The base is 4 rd. longer than the altitude. Find the base and altitude.



4. An inch square is cut out of each corner of a square piece of tin. The sides are turned up to make a box. How long was each side of the square before the corners were cut out if the box has a volume of 64 cu. in.?

5. Two consecutive odd numbers are multiplied together. Their product is 255 sq. ft. Find the numbers.

6. The product of two numbers is 40. One number is 3 more than the other. Find the numbers.



7. A boy draws a square. He adds two inches to two adjacent sides. Then he completes the new square which has an area of 36 sq. in. How long was a side of the first square?

8. The length of a rectangle is 4 in. more than the width. The area is 60 sq. in. Find the length and width.

9. The hypotenuse of a right triangle is 10 in. The base is 2 in. more than the altitude. Find the length of each leg.

10. The pupils in a mathematics club decided to serve refreshments after one of their programs. The expense was \$3. The number of cents each member paid was 20 less than the number in the club. Find the number in the club.

11. A guy wire is 20 ft. long. How far up a pole must it be placed if this distance is to be 4 ft. less than the distance from the foot of the pole to the place where the other end of the wire is fastened?

12. A rectangular flower bed is 12 ft. long and 8 ft. wide. A concrete walk containing an area of 96 sq. ft. is made around it. Find the width of the walk.

VIII. Practice Exercises in Factoring Complete Quadratics

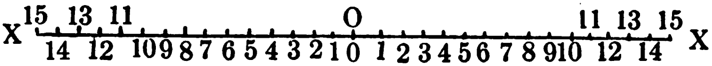
1. $x^2 + 7x + 10 = 0$	**11.** $v^2 - 14v = -48$
2. $x^2 + 17x + 72 = 0$	**12.** $x^2 + 90 = 19x$
3. $y^2 + 4y - 12 = 0$	**13.** $3x^2 + 14x + 8 = 0$
4. $x^2 + 7x - 30 = 0$	**14.** $40a^2 + 33a - 18 = 0$
5. $a^2 - a - 12 = 0$	**15.** $6t^2 - 7t = 10$
6. $x^2 - 3x - 28 = 0$	**16.** $16v^2 - 10v = 21$
7. $x^2 - 6x - 55 = 0$	**17.** $8b^2 + 24 = 28b$
8. $x^2 - 11x + 24 = 0$	**18.** $\frac{3}{2}x^2 - \frac{7}{2}x - 3 = 0$
9. $n^2 - 22n + 120 = 0$	**19.** $.8x^2 + 1.6x + .6 = 0$
10. $x^2 - 5x = 24$	**20.** $56x^2 + 37x - 45 = 0$

CHAPTER FOUR

SUBTRACTION AND DIVISION

A. SUBTRACTION OF POSITIVE NUMBERS

In elementary arithmetic you learned how to subtract one positive number from another when the subtrahend was less than the minuend. Then it seemed to you impossible to subtract 12 from 8. But since you have known about the enlarged number scale, you have really been subtracting such a number as 12 from 8.



1. In what direction must you count to *add* 12 to 8? What is the sum?

2. To subtract we must count in the opposite direction, for subtraction is the opposite of addition. Subtract 12 from 8 by starting at 8 and counting 12 to the left.

(a) What is the difference between 12 and 8?

(b) Add negative 12 to positive 8. What is the sum?

Compare the two results.

${}^{+}8 + {}^{-}12 = {}^{-}4$ Adding a negative number.

${}^{+}8 - {}^{+}12 = {}^{-}4$ Subtracting a positive number.

3. (a) Subtract ${}^{+}12$ from ${}^{-}8$ by counting from ${}^{-}8$ twelve places to the left. What is the difference?

(b) Add ${}^{-}12$ to ${}^{-}8$; what is the sum?

(c) Compare the two results.

${}^{-}8 + {}^{-}12 = {}^{-}20$ Adding a negative number.

${}^{-}8 - {}^{+}12 = {}^{-}20$ Subtracting a positive number.

4. We see that subtracting a positive number is not new, for it is the same as adding a like negative number.

5. To subtract 7 from 12, one must find the number to be added to 7 to make 12. Therefore subtraction may be checked by adding the difference to the subtrahend. If the sum is the minuend, the subtraction is correct. This check holds with signed numbers as you have found that it does with arithmetic numbers.

6. Subtract and check the following:

(a)	(b)	(c)	(d)	(e)
8	$^{-}8$	4	$^{-}4$	$5a$
4	4	8	8	$2a$

(f)	(g)	(h)	(i)	(j)
$^{-}5a$	$2a$	$^{-}2a$	$5ax$	$^{-}5x^2$
$2a$	$5a$	$5a$	$7ax$	$7x^2$

B. SUBTRACTION OF NEGATIVE NUMBERS

I. Meaning of Subtraction of Signed Numbers

1. If the temperature at noon was 15° above zero ($^{+}15°$) and at midnight was 5° below zero ($^{-}5°$), how much warmer was it at noon than at midnight?

(a) To find the difference in temperature, begin at 5° below zero ($^{-}5°$) and count up to $^{+}15°$.

You must count up to 5° to reach 0° and 15° more to reach 15° above zero. So it was 20° warmer at noon.

That is, $^{+}15° - {}^{-}5° = {}^{+}15° + 5°$, or $^{+}5° + 15°$,
$= 20°$

Or, in vertical form,

$$\begin{array}{r} 15° \\ ^{-}5° \\ \hline ^{+}20° \end{array}\text{, difference.} \qquad\qquad \begin{array}{r} \textit{Check} \\ ^{-}5 \\ ^{+}20 \\ \hline ^{+}15 \end{array}\text{, sum.}$$

2. Early in the morning the temperature was $^{-}12°$. At noon it was $^{-}4°$. How much warmer was it at noon than in the morning?

(a) By counting from $^{-}12°$ to $^{-}4°$, you count 8° upward, that is, 8° in the positive direction. Therefore,

$$\begin{aligned} ^{-}4° - {}^{-}12° &= {}^{-}4° + 12° \\ &= {}^{+}8°, \text{ the difference.} \end{aligned}$$

In vertical form,

$$\begin{array}{r} ^{-}4 \\ \underline{^{-}12} \\ ^{+}8, \text{ difference.} \end{array} \qquad\qquad \begin{array}{r} \textit{Check} \\ ^{-}12 \\ \underline{^{+}8} \\ ^{-}4, \text{ sum.} \end{array}$$

3. We see that *to subtract* a *negative* number one may *add* a *like positive number.*

4. A man may increase the money he has in two ways: First, he may earn more money, thus adding to his income. Second, he may cut down on his expenses.

(a) The first is adding a positive amount, the second is subtracting a negative amount. But either way increases his net income.

(b) What other illustrations can you give to show that subtracting a negative number is the same as adding a like positive number?

5. Paris is 2° east longitude. New York is 74° west longitude. How many degrees east of New York is Paris?

6. Point Gallinas, the most northern point of South America, is 13° north latitude. Cape Horn is 55° south latitude. How many degrees long is the continent of South America?

7. Archimedes was a famous inventor and mathematician of Syracuse. He was born 287 B.C. and died in 212 B.C. How long did he live? (If time A.D. is positive, time B.C. is negative.)

8. In Roman history the reign of Caesar Augustus is known as the golden age of the Roman Empire. It began in 31 B.C. and ended with Caesar's death in 14 A.D. How long did it last?

II. Practice in Subtraction of Signed Numbers

Subtract the following numbers. Practice reading the differences until satisfactory speed is developed.

	(a)	(b)	(c)	(d)	(e)
1.	$\begin{array}{r} 14 \\ \underline{6} \end{array}$	$\begin{array}{r} ^{+}6 \\ \underline{^{-}2} \end{array}$	$\begin{array}{r} ^{-}9 \\ \underline{^{+}3} \end{array}$	$\begin{array}{r} ^{-}6 \\ \underline{^{+}10} \end{array}$	$\begin{array}{r} ^{+}8 \\ \underline{^{-}4} \end{array}$
2.	$\begin{array}{r} 12n \\ \underline{7n} \end{array}$	$\begin{array}{r} 3a \\ \underline{^{-}7a} \end{array}$	$\begin{array}{r} ^{-}11b \\ \underline{^{-}2b} \end{array}$	$\begin{array}{r} 17x \\ \underline{^{-}6x} \end{array}$	$\begin{array}{r} ^{-}30y \\ \underline{^{+}6y} \end{array}$
3.	$\begin{array}{r} ^{-}8xy \\ \underline{^{-}2xy} \end{array}$	$\begin{array}{r} ^{+}4ab \\ \underline{^{-}6ab} \end{array}$	$\begin{array}{r} ^{-}9mn \\ \underline{^{+}6mn} \end{array}$	$\begin{array}{r} ^{+}15tu \\ \underline{^{+}3tu} \end{array}$	$\begin{array}{r} ^{-}6xyz \\ \underline{^{+}2xyz} \end{array}$
4.	$\begin{array}{r} 18x \\ \underline{^{-}16x} \end{array}$	$\begin{array}{r} ^{-}40y \\ \underline{^{+}2y} \end{array}$	$\begin{array}{r} ^{-}131 \\ \underline{^{-}71} \end{array}$	$\begin{array}{r} 25xy \\ \underline{^{-}16xy} \end{array}$	$\begin{array}{r} ^{-}25xy \\ \underline{^{+}16xy} \end{array}$

5. $\begin{array}{r} 8x + 6y \\ \underline{2x + 3y} \end{array}$

6. $\begin{array}{r} 2a - 6b \\ \underline{8a + 4b} \end{array}$

7. $\begin{array}{r} 10t + 7u \\ \underline{^{-}3t + 10u} \end{array}$

8. $\begin{array}{r} 8x^2y + 15z \\ \underline{7x^2y - 2z} \end{array}$

9. $\begin{array}{r} ^{-}17ab - 3cd \\ \underline{^{-}4ab - 2cd} \end{array}$

10. $\begin{array}{r} ^{-}4m - 18n \\ \underline{^{-}6m + 30n} \end{array}$

11. $\begin{array}{r} 8x - 3y + 2z \\ \underline{^{-}2x + 6y - 8z} \end{array}$

12. $\begin{array}{r} 14k + 61 - 9m \\ \underline{4k - 81 - 9m} \end{array}$

13. $\begin{array}{r} 9a - 3b + 4c \\ \underline{9a + 3b - 8c} \end{array}$

14. $\begin{array}{r} 6b - 7c + 4d \\ \underline{^{-}4b + 2c - 8d} \end{array}$

15. $\begin{array}{r} 8x + 9y + 6z \\ \underline{^{-}10x + 2y - 3z} \end{array}$

16. $\begin{array}{r} 4a - 5x + 6y \\ \underline{^{-}5a - 7x - 2y} \end{array}$

C. DIVISION OF SIGNED NUMBERS

I. Comparison with Multiplication

1. (a) If $(^+4)(^+6) = {}^+24$, then $^+24 \div {}^+4 = {}^+6$, or $\dfrac{+24}{+4} = +6$

(b) If $(^+4)(^-6) = {}^-24$, then $^-24 \div {}^+4 = {}^-6$, or $\dfrac{-24}{+4} = {}^-6$

(c) If $(^-4)(^+6) = {}^-24$, then $^-24 \div {}^-4 = {}^+6$, or $\dfrac{-24}{-4} = +6$

(d) If $(^-4)(^-6) = {}^+24$, then $^+24 \div {}^-4 = {}^-6$, or $\dfrac{+24}{-4} = -6$

2. Since division is so closely related to multiplication, the same laws of signs apply to both processes.

(a) If the signs of the dividend and divisor are both *plus*, what is the sign of the quotient?
(b) If the signs of the dividend and divisor are both minus, what is the sign of the quotient?
(c) How can you answer both of these questions in one statement?
(d) Under what conditions is the quotient negative?

II. Practice Exercises with Arithmetic Numbers

Read the following quotients:

	(a)	(b)	(c)	(d)
1.	$^+30 \div {}^+5 =$	$^-30 \div {}^-5 =$	$^+30 \div {}^-5 =$	$^-30 \div {}^+5 =$
2.	$^-42 \div {}^-7 =$	$^+64 \div {}^-8 =$	$^-100 \div {}^+10 =$	$^+96 \div {}^+12 =$
3.	$^-144 \div {}^+12 =$	$^+81 \div {}^+9 =$	$^+36 \div {}^-4 =$	$^-18 \div {}^-9 =$
4.	$^+32 \div {}^-4 =$	$^+16 \div {}^-2 =$	$^-32 \div {}^-8 =$	$^-90 \div {}^+10 =$

III. Division by Monomials

Divide $35x^2$ by $5x$. Explain how you find the coefficient of the quotient and how you find the exponent of the letter.

Divide the following:

	(a)	(b)	(c)
1.	$\dfrac{48m^2}{-6m}$	$\dfrac{-39xy}{3x}$	$\dfrac{54ab}{6a}$
2.	$\dfrac{-14n^2}{2n}$	$\dfrac{52cd}{-4d}$	$\dfrac{-32tu}{-4t}$
3.	$\dfrac{56a^3}{4a}$	$\dfrac{-18a^2b}{+2ab}$	$\dfrac{+24m^2n}{-3m}$
4.	$\dfrac{6mn-3n}{3n}$	$\dfrac{4ab-2b}{2b}$	$\dfrac{36xy+4x}{4x}$
5.	$\dfrac{-21xy+30x^2}{3x}$	$\dfrac{21m^2n-3mn^2}{3mn}$	$\dfrac{8a+2b}{2}$
6.	$\dfrac{56xy^2-8xy^3}{8xy^2}$	$\dfrac{90x^3-30x^2}{-30x^2}$	$\dfrac{72a^3b+30a^2b^2}{6a^2b}$

IV. Division by Polynomials

1. If the area of a rectangle is $12x^2 - 41xy + 35y^2$ and one dimension is $3x - 5y$, what is the other dimension?

(a) How can you find the length if the area and width are known?

(b) *Solution:*

Given $S_{\square} = 12x^2 - 41xy + 35y^2$

$w = 3x - 5y$

To find l.

$$l = \frac{S}{w}$$

$$\begin{array}{l|l}
12x^2 - 41xy + 35y^2 & 3x - 5y \\ \cline{2-2}
\underline{12x^2 - 20xy} & \underline{4x - 7y} \\
\quad\;\; - 21xy + 35y^2 & \\
\quad\;\; \underline{- 21xy + 35y^2} &
\end{array}$$

$\therefore l = 4x - 7y$

(c) *Proof:* $(3x - 5y)(4x - 7y) = 12x^2 - 41xy + 35y^2$

2. (a) In division, in what order must the terms of the dividend and divisor be arranged?
(b) How is the first term of the quotient found?
(c) Explain how the remainder, $-21xy$, is obtained in the first subtraction.

3. The volume of a rectangular prism is $60m^3 - 19m^2n - 32mn^2 + 12n^3$ cu. in. If the height is $3m - 2n$ inches, what is the area of the base?

(a) The volume is the product of what three dimensions? Having given the volume and one dimension how may you find the area of the base?

(b) *Solution:*

Formula, $V_{pr} = lwh$

$$\frac{V}{h} = lw, \text{ or base.}$$

$$\begin{array}{llll|l}
60m^3 & -\ 19m^2n & -\ 32mn^2 & +\ 12n^3 & 3m - 2n \\
\underline{60m^3} & \underline{-\ 40m^2n} & & & \overline{20m^2 + 7mn - 6n^2} \\
 & +\ 21m^2n & -\ 32mn^2 & & \\
 & \underline{+\ 21m^2n} & \underline{-\ 14mn^2} & & \\
 & & -\ 18mn^2 & +\ 12n^3 & \\
 & & \underline{-\ 18mn^2} & \underline{+\ 12n^3} &
\end{array}$$

$\therefore\ 20n^2 + 7mn - 6n^2 = lw$, or base.

(c) Find the other two dimensions by factoring the area of the base.

(d) Check division and factoring if $m = 3$ and $n = 2$.

$$V = lwh$$

$$\begin{array}{l}
60m^3 - 19m^2n - 32mn^2 + 12n^3 = (4m+3n)(5m-2n)(3m-2n) \\
60\cdot 3^3 - 19\cdot 3^2\cdot 2 - 32\cdot 3\cdot 2^2 + 12\cdot 2^3 = (4\cdot 3+3\cdot 2)(5\cdot 3-2\cdot 2)(3\cdot 3-2\cdot 2) \\
60\cdot 27 - 19\cdot 9\cdot 2 - 32\cdot 3\cdot 4 + 12\cdot 8 = (12+6)(15-4)(9-4) \\
1620 - 342 - 384 + 96 = 18\cdot 11\cdot 5 \\
990 = 990
\end{array}$$

4. If the volume and base of a prism are known, how may the height be found?

Given, $V = 60m^3 - 19m^2n - 32mn^2 + 12n^3$

Base, or $lw = 20m^2 + 7mn - 6n^2$

To find h,

$$h = \frac{V}{lw}$$

$$\begin{array}{rrrr|l}
60m^3 & - 19m^2n & - 32mn^2 & + 12n^3 & 20m^2 + 7mn - 6n^2 \\
\underline{60m^3} & \underline{+ 21m^2n} & \underline{- 18mn^2} & & \overline{3m - 2n} \\
 & - 40m^2n & - 14mn^2 & + 12n^3 & \\
 & \underline{- 40m^2n} & \underline{- 14mn^2} & \underline{+ 12n^3} &
\end{array}$$

5. Division may be checked,
   (a) by numerical substitution,
   (b) by division, using quotient as the divisor,
   (c) by multiplying divisor by quotient.

6. (a) Divide $8x^3 + 12x^2y + 6xy^2 + y^3$ by $2x + y$.
   (b) Factor the quotient.
   (c) Check in three ways.

7. (a) Divide $6a^3 + 7a^3b - 16ab^2 - 12b^3$ by $3a + 2b$.
   (b) Factor the quotient.
   (c) Check in three ways.

V. Practice Exercises

1. (a) From the given volumes and heights or areas of bases of prisms, find the areas of the bases or heights.
   (b) Factor the areas of the bases to find the length and width.
   (c) Write the volume as equal to the product of its three factors.
   (d) Check the results from the given values.

No.	Volume	Height	Area of Base	Length	Width	Check
(1)	$x^3 + x^2y - 14xy^2 - 24y^3$	$x - 4y$	....	....	....	If $x = 7, y = 1$
(2)	$6x^3 - 11x^2y - 26xy^2 + 40y^3$	$3x - 4y$	....	....	....	$x = 6, y = 2$
(3)	$24a^3 - 14a^2b - 11ab^2 + 6b^3$	....	$8a^2 - 10ab + 3b^2$	....	....	$a = 5, b = 3$
(4)	$12c^3 + 13c^2d - 21cd^2 - 6d^3$	....	$3c^2 + cd - 2d^2$	....	....	$c = 4, d = 2$
(5)	$a^3 - a^2b - ab^2 - b^3$	$a - b$	....	....	....	$a = 8, b = 3$
(6)	$4l^3 + 12l^2k - lk^2 - 15k^3$	....	$4l^2 + 16lk + 15k^2$	....	....	$l = 3, k = 2$
(7)	$30l^3 - 31l^2w - 15lw^2 + 4w^3$	$2l + w$	....	....	....	$l = 2, w = 1$
(8)	$x^3 + x^2y - 14xy^2 - 24y^3$	....	$x^2 + 5xy + 6y^2$	....	....	$x = 9, y = 2$
(9)	$24a^3 - 2a^2b - 5ab^2 + b^3$	$4a - b$	....	....	....	$a = 12, b = 5$
(10)	$6r^3 + 7r^2s - 11rs^2 - 12s^3$	....	$2r^2 + 5rs + 3s^2$	....	....	$r = 6, s = 3$
(11)	$24x^3 - 58x^2y + 37xy^2 - 6y^3$	$2x - 3y$	....	....	....	$x = 8, y = 2$
(12)	$5x^3 - 22x^2y + 15xy^2 + 18y^3$	....	$5x^2 - 7xy - 6y^3$	....	....	$x = 10, y = 2$
(13)	$8m^3 + 6m^2n - 9mn^2 - 5n^3$	$2m + n$	....	....	....	$m = 7, n = 3$
(14)	$36a^3 - 12a^2b - 11ab^2 + 2b^3$	....	$12a^2 + 4ab - {}^2b$	....	....	$a = 9, b = 4$
(15)	$64x^3 - 216x^2y - 58xy^2 + 105y^3$	$8x - 5y$	....	....	....	$x = 5, y = 1$
(16)	$15a^3 + 29a^2b + 6ab^2 - 8b^3$	....	$5a^2 + 3ab - 2b^2$	....	....	$a = 10, b = 3$

VI. Order of Terms in Division

1. (a) You have learned to factor $a^2 - b^2$ into two binomials, $a + b$ and $a - b$.
   (b) By division show that the two factors are correct.
   (c) Since the product of two binomials regularly gives a trinomial product, one of the terms of this product must be zero.

   (d) As a trinomial, $a^2 - b^2 = a^2 + 0 - b^2$.
   (e) Divide $a^2 - b^2$ by $a + b$.

$$\begin{array}{l|l} a^2 + 0 - b^2 & a + b \\ \underline{a^2 + ab} & \overline{a - b} \\ \quad - ab - b^2 & \\ \quad \underline{- ab - b^2} & \end{array}$$

2. Divide $a^2 - b^2$ by $a - b$.
3. Divide $9x^2 - 4y^2$ by $3x + 2y$.
4. Divide $25x^2 - 16y^2$ by one of its factors.
5. Divide each of the following by one of its factors.

   (a) $64m^2 - 36n^2$.
   (b) $49a^2 - 81x^2$.
   (c) $\frac{1}{4}a^2 - \frac{1}{9}b^2$.
   (d) $\frac{1}{25}c^2 - \frac{1}{16}d^2$.
   (e) $\frac{1}{100}x^2 - \frac{1}{81}y^2$.
6. (a) Find the product of $(5a + b)(3a + 4b)(a - 3b)$.
   (b) How many terms are in the product?
   (c) What is the highest power of a in the product? What other powers of a are in the other terms? Is there any term without a? What is the power of b in that term?
   (d) If a product has a third power of a letter in it and no higher power, how many terms do you expect it to have?
   (e) If such a product has less than four terms one or more of them must be zero.

(f) In the complete product, how does the exponent of the first letter decrease?

(g) In which term does the second letter appear?

(h) How does its exponent change?

(i) In an incomplete product how can you know which term is omitted?

7. (a) Which term is omitted in $x^3 - 19x + 30$? Write it as a complete expression.

(b) Divide $x^3 - 19x + 30$ by $x - 2$.

(c) Factor the quotient.

(d) Give three factors of $x^3 - 19x + 30$.

(e) Test the factoring by checking with $x = 3$.

NOTE. — If you try to check with $x = 2$, the value of both dividend and divisor will be zero.

$$\frac{x^3 - 19x + 30}{x - 2} = x^2 + 2x - 15$$

$$\frac{8 - 38 + 30}{2 - 2} = 4 + 4 - 15$$

$$\frac{0}{0} = -7$$

Whenever the number used for checking makes the divisor zero, it cannot be a real check.

8. (a) What terms are omitted in the expression $a^3 + b^3$?

(b) Divide $a^3 + b^3$ by $a + b$.

$$\begin{array}{l|l}
a^3 + 0 + 0 + b^3 & a + b \\
\underline{a^3 + a^2b} & \overline{a^2 - ab + b^2} \\
\quad - a^2b + 0 & \\
\quad \underline{- a^2b - ab^2} & \\
\qquad\qquad ab^2 + b^3 & \\
\qquad\qquad \underline{ab^2 + b^3} &
\end{array}$$

(c) Check with $a = 5$ and $b = 3$.

(d) Check by dividing $a^3 + b^3$ by $a^2 - ab + b^2$.

(e) Check by multiplying $a^2 - ab + b^2$ by $a + b$.

9. (a) Make the divisions indicated in the following.

(b) Check the first problem by numerical substitution; the second by reversed division; the third by multiplication. Continue the checks in the same order.

(c) Whenever possible, factor the quotient.

No.	Dividend	Divisor	Quotient	Factors
(1)	$a^3 - b^3$	$a - b$	....	....
(2)	$8x^3 + 27y^3$	$2x + 3y$	....	....
(3)	$a^3 - 125$	$a - 5$	....	....
(4)	$64m^3 + 125$	$4m + 5$	....	....
(5)	$x^3 - 7x + 6$	$x - 1$	....	....
(6)	$x^3 + y^3$	$x + y$	....	....
(7)	$27a^3 - 64$	$3a - 4$	....	....
(8)	$125c^3 - 216d^3$	$5c - 6d$	....	....
(9)	$x^3 - 12x + 16$	$x - 2$	....	....
(10)	$x^3 - 13x - 12$	$x - 4$	....	....
(11)	$8n^3 - 64t^3$	$2n - 4t$	....	....
(12)	$64b^3 - 343c^3$	$4b - 7c$	....	....
(13)	$x^3 - 28x + 48$	$x + 6$	....	....
(14)	$8x^3 - 125y^3$	$2x - 5y$	....	....
(15)	$b^3 - 216$	$b - 6$	....	....
(16)	$x^3 - 49x + 120$	$x + 8$	....	....
(17)	$d^3 - 512$	$d - 8$	....	....
(18)	$125a^3 + 64b^3$	$5a + 4b$	....	....
(19)	$8t^3 - 729u^3$	$8t - 9u$	....	....
(20)	$x^3 - 39x + 70$	$x - 5$	....	....

CHAPTER FIVE

EQUATIONS WITH PARENTHESES

A. PARENTHESES WITH COEFFICIENTS

I. Derived from Problems

1. (a) Draw an unmeasured line x inches long.

(b) Draw a square $5x + 1$ inches long.

(c) Draw an equilateral triangle whose side is $4x - 3$ inches.

(d) The difference between the perimeters of the square and triangle is 69 inches.

(e) How long is the unmeasured line?

(f) Find the side of the square and its perimeter.

(g) Find the side of the triangle and its perimeter.

(h) *Solution:*

$$P_{\square} = 4e \qquad\qquad P_{\triangle} = 3a$$
$$= 4(5x + 1) \qquad\qquad = 3(4x - 3)$$
$$P_{\square} - P_{\triangle} = 4(5x + 1) - 3(4x - 3)$$
$$P_{\square} - P_{\triangle} = 69$$
$$4(5x + 1) - 3(4x - 3) = 69$$

(1) You have already learned how to multiply an expression in parenthesis as $(5x + 1)$ by the factor 4. After the multiplication is done, the parenthesis is no longer needed.

(2) If a quantity within a parenthesis has a negative factor with it, as the -3 with $(4x - 3)$, the $4x - 3$ is to be multiplied in the same way by -3 and the parenthesis dropped.

(3) Performing such indicated multiplication, or removing the parenthesis, makes the equation

$$4(5x + 1) - 3(4x - 3) = 69$$

become $$20x + 4 - 12x + 9 = 69.$$

Combining terms gives

$8x + 13 = 69.$

Transposing 13 gives

$$\begin{aligned} 8x &= 69 - 13 \qquad \text{Why?} \\ 8x &= 56 \\ x &= 7, \text{ number of inches in given line. Why?} \\ 5x + 1 &= 35 + 1 \\ &= 36, \text{ number of inches in side of square.} \\ 4x - 3 &= 28 - 3 \\ &= 25, \text{ number of inches in side of triangle.} \\ P_{\square} &= 4 \cdot 36 = 144 \\ P_{\triangle} &= 3 \cdot 25 = 75 \end{aligned}$$

(4) *Proof:* $P_{\square} - P_{\triangle} = 144 - 75$
$69 = 69$

II. Practice Exercise in Removing Parentheses with Coefficients

1. $5(4x - 1) - 4(3x - 2) = 67$
2. $5(6x - 5) - 2(3x - 2) = 89$
3. $7(3x + 2) - 8(2x - 3) = 73$
4. $3(x + 4) + 7(-6x + 1) = -20$
5. $8(5x + 3) - 5(2x + 3) = 69$
6. $6(3a + 4) - 5(-4a - 1) = -9$
7. $8(4x - 3) - 3(3x - 4) = 34$
8. $4(7x - 8) - 5(2x - 7) = 115$
9. $20(x + 3) - 9(x + 8) = 87$

10. $4(4x - 3) + 5(-3x - 8) = -55$
11. $8(2x - 1) - 3(-x - 3) = 20$
12. $5(3x - 8) + \frac{1}{2}(2x + 9) = -3\frac{1}{2}$
13. $7(7x - 5) - 5(9x - 2) = 2$
14. $4(10x - 9) - 3(8x - 7) = 14$
15. $5(x - 7) + 4(x - 6) = 22$
16. $7(3x + 8) - 3(6x + 13) = 57$
17. $14(b - 6) - \frac{1}{3}(b - 9) = 1$
18. $8(3m - 2) - 9(4m - 5) = 5$
19. $4(5c - 4) - \frac{1}{3}(3c - 7) = 5$
20. $4(6t - 5) - 3(9t - 8) = 29$

B. PARENTHESES WITHOUT COEFFICIENTS

I. Derived from Problems

1. (a) Draw an unmeasured line n inches long.
(b) Draw a square on a line $3n + 5$ inches long.
(c) Draw another square on a line $2n - 1$ inches long.
(d) The difference between the perimeters of these two squares is 40 inches.
(e) How long is the unmeasured line?
Find the side of each square.
(f) Find the perimeter of each square.
(g) *Solution:*

$$P_{\square} = 4e$$
$$P_{\boxed{2}} = 4(3n + 5)$$
$$P_{\boxed{2}} = 4(2n - 1)$$
$$P_{\boxed{2}} - P_{\boxed{2}} = 4(3n + 5) - 4(2n - 1)$$
$$P_{\boxed{2}} - P_{\boxed{2}} = 40$$
$$4(3n + 5) - 4(2n - 1) = 40$$

(1) This equation may be solved by performing the multiplication as indicated. But there is a shorter way. It is easy to see that 4 is a factor of each expression or term of the.

equation; therefore, both members of the equation may be divided by 4, thus:

$$(3n + 5) - (2n - 1) = 10$$

(2) The first parenthesis may be dropped because it is positive.

(3) The second expression was $-4(2n - 1)$. When this is divided by 4, as $\frac{-4(2n-1)}{4}$, it becomes $-1(2n - 1)$.

(4) Just as the positive 1 is understood before the $(3n + 5)$ so the -1 is understood but not written before $-(2n - 1)$. But to remove the parentheses, the expression $2n - 1$ must be multiplied by -1.

Therefore, $(3n + 5) - (2n - 1) = 10$

becomes $3n + 5 - 2n + 1 = 10$

Combining terms gives, $n + 6 = 10$

$$n = 10 - 6$$

$n = 4$ in., length of line.

$$3n + 5 = 12 + 5$$

$= 17$ in., side of $\boxed{1}$.

$$2n - 1 = 8 - 1$$

$= 7$ in., side of $\boxed{2}$.

$$\therefore P_{\boxed{1}} = 4 \cdot 17 = 68$$

$$\text{and } P_{\boxed{2}} = 4 \cdot 7 = 28$$

(h) *Proof:*

$$P_{\boxed{1}} - P_{\boxed{2}} = 40$$

$$68 - 28 = 40$$

$$40 = 40$$

II. Practice Exercises in Removing Parentheses without Coefficients

1. $(4n + 7) - (3n - 1) = 12$

2. $(5x - 2) + (3x - 7) = 5$

3. $(6a - 5) - (2a + 7) = 8$

4. $(3b - 4) + (5b - 8) = 4$
5. $(7x - 2) - (-2x + 6) = 10$
6. $(-2x + 5) + (3x + 4) = -1$
7. $(5c + 3) - (4c - 9) = 6$
8. $(8b - 3) + (6b + 7) = 11$
9. $(4m - 3) - (2m - 9) = 4$
10. $(10y - 7) - (-2y - 5) = -8$
11. $(4x + 5) + (-3x - 7) = -8$
12. $(6t - 11) - (-2t + 5) = -32$
13. $(14x + 3) - (9x - 5) = 23$
14. $(12x - 1) - (-3x - 5) = 30$
15. $(8a - 3) - (-4a - 3) = 6$
16. $(16a + 5) - (13a + 6) = -2$
17. $(20x + 7) - (4x - 9) = 20$
18. $(-3t - 11) - (-7\mathrm{t} - 9) = 18$
19. $(9x - 2) - (7x + 5) = 2$
20. $(7x - 4) - (-5x + 9) = -7$

CHAPTER SIX

FRACTIONS

A. ADDITION AND SUBTRACTION OF FRACTIONS

I. With Denominators Prime to Each Other

1. To add $\frac{1}{2}$ to $\frac{1}{3}$, you learned to take the sum of 2 and 3 for the numerator and the product of 2 and 3 for the denominator of the sum.

$$\frac{1}{2} + \frac{1}{3} = \frac{3+2}{2 \cdot 3} = \frac{5}{6}.$$

(a) To add fractions in this way, what must be the numerator of each?

(b) Is it necessary that the two denominators be prime to each other?

2. To show that any two fractions like these may be added thus, let $\frac{1}{a}$ and $\frac{1}{b}$ be the fractions. What is their sum?

(a) What is the common denominator for $\frac{1}{2}$ and $\frac{1}{3}$? For $\frac{1}{a}$ and $\frac{1}{b}$?

(b) How may you change $\frac{1}{2}$ and $\frac{1}{3}$ to sixths? How may you change $\frac{1}{a}$ and $\frac{1}{b}$ to their common denominator, ab?

$\frac{1}{2} = \frac{3}{6}$	$\frac{1}{a} = \frac{b}{ab}$
$\frac{1}{3} = \frac{2}{6}$	$\frac{1}{b} = \frac{a}{ab}$
$\frac{1}{2} + \frac{1}{3} = \frac{3+2}{6}$	$\frac{1}{a} + \frac{1}{b} = \frac{b+a}{ab}$
$= \frac{5}{6}$	or $\frac{a+b}{ab}$

(c) When you solve the general problem $\frac{1}{a} + \frac{1}{b} = \frac{a+b}{ab}$, you have found the sum for all such pairs of fractions.

3. Give the following sums:

	(1)	(2)	(3)
(a)	$\frac{1}{3} + \frac{1}{4}$	$\frac{1}{x} + \frac{1}{y}$	$\frac{1}{3x} + \frac{1}{4y}$
(b)	$\frac{1}{3} + \frac{1}{5}$	$\frac{1}{c} + \frac{1}{d}$	$\frac{1}{3c} + \frac{1}{5d}$
(c)	$\frac{1}{5} + \frac{1}{8}$	$\frac{1}{m} + \frac{1}{n}$	$\frac{1}{5m} + \frac{1}{8n}$
(d)	$\frac{1}{5} + \frac{1}{6}$	$\frac{1}{a} + \frac{1}{x}$	$\frac{1}{5a} + \frac{1}{6x}$
(e)	$\frac{1}{5} + \frac{1}{7}$	$\frac{1}{p} + \frac{1}{q}$	$\frac{1}{5p} + \frac{1}{7q}$

4. If the denominators are binomials that are prime to each other, the sum of two fractions may be found in the same way.

$$\frac{1}{x+y} + \frac{1}{x-y} = \frac{x+y+x-y}{(x+y)(x-y)}$$
$$= \frac{2x}{x^2 - y^2}$$

II. Practice Exercises

Find the sums of:

1. $\frac{1}{2a+b} + \frac{1}{2a-b}$

2. $\frac{1}{2x-3b} + \frac{1}{3x-2b}$

3. $\frac{1}{c+4d} + \frac{1}{c-3d}$

4. $\frac{1}{2a-7} + \frac{1}{a+5}$

5. $\frac{1}{3a - 4b} + \frac{1}{4a - 3b}$

13. $\frac{1}{y - 8z} + \frac{1}{8y + z}$

6. $\frac{1}{x - 8y} + \frac{1}{9x - 2y}$

14. $\frac{1}{6m + n} + \frac{1}{7m - n}$

7. $\frac{1}{6m - 5n} + \frac{1}{3m + 2n}$

15. $\frac{1}{4a - b} + \frac{1}{6a - 5b}$

8. $\frac{1}{4x - y} + \frac{1}{4x + y}$

16. $\frac{1}{x - 4y} + \frac{1}{x - 2y}$

9. $\frac{1}{2b - 9} + \frac{1}{6b + 7}$

17. $\frac{1}{10c + 3d} + \frac{1}{3c - 2d}$

10. $\frac{1}{7c + 2d} + \frac{1}{6c - d}$

18. $\frac{1}{5a - 2d} + \frac{1}{6a + d}$

11. $\frac{1}{5x - 9y} + \frac{1}{2x + 11y}$

19. $\frac{1}{8x + 9z} + \frac{1}{2x - 7z}$

12. $\frac{1}{3r - 5s} + \frac{1}{3r + 5s}$

20. $\frac{1}{5x - 5y} + \frac{1}{x + y}$

III. With Denominators that are not Prime

1. (a) Add the fractions $\frac{3}{8}$ and $\frac{5}{18}$.
   (b) What must be done to these two fractions before you can add them? Why?
   (c) What is the lowest common denominator of eighths and eighteenths? How did you find it?
   (d) What is the meaning of each word in the expression *lowest common denominator?*
   (e) How do you change each fraction to the new denominator?
   (f) If the denominators of $\frac{3}{8}$ and $\frac{5}{18}$ are factored, the fractions become $\frac{3}{2^3}$ and $\frac{5}{2 \cdot 3^2}$.

(g) What two prime factors must the common denominator have? How many 2's must it have? How many 3's? Why?

2. (a) In these fractions, let x take the place of the factor 2 in the denominators, and y take the place of the factor 3. Then $\frac{3}{2^3}$ becomes $\frac{3}{x^3}$ and $\frac{5}{2 \cdot 3^2}$ becomes $\frac{5}{xy^2}$.

(b) To add $\frac{3}{x^3}$ to $\frac{5}{xy^2}$, we must find the lowest common denominator of x^3 and xy^2. Evidently this denominator must contain the two factors x and y. How many times must x be used as a factor in order that x^3 may be divided into it? How many times must y be used as a factor in order that the common denominator may be divisible by xy^2?

(c) Evidently the lowest common denominator of these fractions is x^3y^2, for it has the lowest powers of x and y that make it divisible by both x^3 and xy^2.

(1) $\frac{3}{8} = \frac{?}{72}$

$$\frac{3}{8} \times \frac{9}{9} = \frac{27}{72}$$

How is the multiplier $\frac{9}{9}$ found?

$$\frac{5}{18} = \frac{?}{72}$$

$$\frac{5}{18} \times \frac{4}{4} = \frac{20}{72}$$

(2) $\frac{3}{x^3} = \frac{?}{x^3y^2}$

$$\frac{3}{x^3} \times \frac{y^2}{y^2} = \frac{3y^2}{x^3y^2}$$

How is the multiplier $\frac{y^2}{y^2}$ found?

$$\frac{5}{xy^2} = \frac{?}{x^3y^2}$$

$$\frac{5}{xy^2} \times \frac{x^2}{x^2} = \frac{5x^2}{x^3y^2}$$

(d) Explain how each change is made.

(e) Add each pair of fractions.

(1) $\frac{3}{8}+\frac{5}{18}=\frac{27}{72}+\frac{20}{72}=\frac{27+20}{72}=\frac{47}{72}$

(2) $\frac{3}{x^3}+\frac{5}{xy^2}=\frac{3y^2}{x^3y^2}+\frac{5x^2}{x^3y^2}=\frac{3y^2+5x^2}{x^3y^2}$ or $\frac{5x^2+3y^2}{x^3y^2}$

3. Subtract $\frac{5}{18}$ from $\frac{3}{8}$ and $\frac{5}{xy^2}$ from $\frac{3}{x^3}$

(a) $\frac{3}{8}-\frac{5}{18}=\frac{27}{72}-\frac{20}{72}=\frac{27-20}{72}=\frac{7}{72}$

(b) $\frac{3}{x^3}-\frac{5}{xy^2}=\frac{3y^2}{x^3y^2}-\frac{5x^2}{x^3y^2}=\frac{3y^2-5x^2}{x^3y^2}$

4. From these examples we see that addition and subtraction of fractions with literal numbers is the same as with arithmetic numbers.

IV. Practice Exercise

Add or subtract the following fractions as indicated:

1. $\frac{2}{3x}+\frac{3}{4y}$

2. $\frac{2}{3x}-\frac{3}{4y}$

3. $\frac{3}{4a}+\frac{2}{5y}$

4. $\frac{3}{4m}-\frac{5}{7n}$

5. $\frac{5}{3c}-\frac{3}{4d}$

6. $\frac{7}{4x}+\frac{3}{5y}$

7. $\frac{4}{a^3}-\frac{6}{ab^2}$

8. $\frac{5}{c^4}-\frac{7}{c^2d^3}$

9. $\frac{4x}{y}-\frac{3y}{x}$

10. $\frac{6ab}{5}+\frac{4ab}{2}$

11. $\frac{7x}{z}+\frac{4x}{y}$

12. $\frac{8}{3x}+\frac{5}{6x}$

13. $\frac{9}{ax} - \frac{3}{a^2x}$

14. $\frac{10}{x^2y} - \frac{8}{xy^2}$

15. $\frac{3}{x^2y^3} - \frac{5}{x^3y}$

16. $\frac{5x}{2y} + \frac{3x}{5y}$

17. $\frac{10a}{3b^2} + \frac{4b}{5a^2}$

18. $\frac{5ab}{8x} - \frac{9ab}{7y}$

V. Sum of a Series of Fractions

1. (a) The sum of a series of fractions may be found provided each fraction and each whole number in the series are reduced to the lowest common denominator.

(b) $\frac{5}{6} + 2 - \frac{4}{9} - \frac{3}{4} = \frac{5}{2 \cdot 3} + 2 - \frac{4}{3^2} - \frac{3}{2^2}$

$\boxed{L.\,C.\,D. = 2^2 \cdot 3^2 = 36}$

$$= \frac{30}{36} + \frac{72}{36} - \frac{16}{36} - \frac{27}{36}$$

$$= \frac{30 + 72 - 16 - 27}{36}$$

$$= \frac{59}{36}$$

$$= 1\frac{23}{36}$$

(c) Explain each step.

2. (a) In like manner the sum of a series of fractions with literal numbers may be found.

(b) $\frac{5}{xy} + 2 - \frac{4}{y^2} - \frac{3}{x^2} = \frac{5xy}{x^2y^2} + \frac{2x^2y^2}{x^2y^2} - \frac{4x^2}{x^2y^2} - \frac{3y^2}{x^2y^2}$

$\boxed{L.\,C.\,D. = x^2y^2}$

$$= \frac{5xy + 2x^2y^2 - 4x^2 - 3y^2}{x^2y^2}$$

(c) Why cannot this sum be simplified?

VI. Practice Exercise

1. $\frac{7}{9} + 3 - \frac{2}{3} - \frac{2}{5}$

2. $\frac{7}{x^2} + 3 - \frac{2}{x} - \frac{2}{y}$

3. $\frac{4}{x^3} - 2 + \frac{3}{y} - \frac{7}{xy}$

4. $5 + \frac{6}{y} - \frac{8}{x^2} - \frac{3}{y^2}$

5. $\frac{9}{x^2y} - \frac{4}{xy} - 3 - \frac{1}{y}$

6. $\frac{7}{ab} - \frac{9}{b^2} - \frac{5}{a} - 3$

7. $\frac{8}{ac^2} - \frac{3}{c} + \frac{1}{a} + 5$

8. $\frac{3}{abx} + \frac{2}{a^2x} - \frac{1}{ax^2} + \frac{5}{b^2}$

9. $\frac{x}{3y} - \frac{2}{5x} - 3 + \frac{5}{xy}$

10. $\frac{7x}{8} - \frac{3y}{2} - 4 - \frac{xy}{16}$

11. $\frac{8}{x^2} - \frac{2}{y} + 7 - \frac{9}{xy^2}$

12. $\frac{3ab}{x} - \frac{2b}{y} - 6 + \frac{a}{x^2}$

13. $\frac{8c}{9d} - \frac{2d}{3c} + 3 - \frac{5cd}{12}$

14. $\frac{7x}{y} - \frac{3y}{x^2} - \frac{2x}{y^2} + 6$

15. $\frac{6}{a^2b^2} - \frac{2}{a^2b} + \frac{3}{ab^2} + 9$

16. $\frac{8}{cd} + 3 - \frac{9}{d^2} - \frac{5}{c^2}$

17. $\frac{4}{a^3} - \frac{5}{a^2} + \frac{2}{a} - 1$

18. $\frac{7}{x^2y} - \frac{8}{y} + \frac{2}{x} - \frac{3}{xy^2}$

19. $\frac{11}{ab} - \frac{2}{a} + \frac{3}{b} - 4$

20. $\frac{14}{ax} - \frac{10}{a^2} + \frac{2}{x^2} - 8$

VII. Fractions with Polynomial Denominators

1. (a) Fractions having binomial or trinomial denominators may be added in the same way as those having monomial denominators.

(b) Find the sum of $\frac{1}{x} - \frac{2}{x^2 + x} + \frac{3}{x^2 - x}$

(c) *Solution:*

$$\frac{1}{x} - \frac{2}{x^2 + x} + \frac{3}{x^2 - x} = \frac{1}{x} - \frac{2}{x(x+1)} + \frac{3}{x(x-1)}$$

$$\boxed{L.\ C.\ D. = x(x+1)(x-1)}$$

$$= \frac{(x+1)\,(x-1)}{x(x+1)\,(x-1)} - \frac{2(x-1)}{x(x+1)\,(x-1)} + \frac{3(x+1)}{x(x+1)\,(x-1)}$$

$$= \frac{(x+1)\,(x-1) - 2(x-1) + 3(x+1)}{x(x+1)\,(x-1)}$$

$$= \frac{x^2 - 1 - 2x + 2 + 3x + 3}{x(x+1)\,(x-1)}$$

$$= \frac{x^2 + x + 4}{x(x+1)\,(x-1)}$$

(d) Can the numerator $x^2 + x + 4$ be factored? If not, the sum may be left as $\dfrac{x^2 + x + 4}{x(x+1)\,(x-1)}$, or the factors of the denominator may be multiplied, as $\dfrac{x^2 + x + 4}{x^3 - x}$.

(e) Always try to factor the numerator of the sum first and if possible reduce the fractions to lowest terms.

(f) How was the lowest common denominator found?

(g) Explain how $\dfrac{1}{x}$ was changed to $\dfrac{(x+1)\,(x-1)}{x(x+1)\,(x-1)}$.

(h) By reducing $\dfrac{(x+1)\,(x-1)}{x(x+1)\,(x-1)}$ to lowest terms, show that it equals $\dfrac{1}{x}$.

(i) Explain each step in the addition.

(j) Throughout the process, it is best to keep the L. C. D. in its factored form. Be sure that the line of the fraction is as long as its terms.

(k) Check the addition with $x = 2$.

$$\frac{1}{x} - \frac{2}{x^2 + x} + \frac{3}{x^2 - x} = \frac{x^2 + x + 4}{x(x + 1)(x - 1)}$$

$$\frac{1}{2} - \frac{2}{4 + 2} + \frac{3}{4 - 2} = \frac{4 + 2 + 4}{2(2 + 1)(2 - 1)}$$

$$\frac{1}{2} - \frac{2}{6} + \frac{3}{2} = \frac{10}{6}$$

$$\frac{1}{2} - \frac{1}{3} + \frac{3}{2} = \frac{5}{3}$$

$$2 - \frac{1}{3} = \frac{5}{3}$$

$$1\frac{2}{3} = 1\frac{2}{3}$$

2. (a) Add the following fractions.

(b) Check each.

(1) $\frac{1}{a} - \frac{3}{a^2 + a} + \frac{4}{a^2 - a}$

(2) $\frac{2}{m} + \frac{3}{m^2 - m} - \frac{5}{m^2 + m}$

(3) $\frac{4}{x - y} - \frac{3}{x^2 - xy} + \frac{5}{x}$

(4) $\frac{3}{x^2 - y^2} - \frac{6}{x - y} + \frac{4}{x + y}$

(5) $\frac{1}{a^2 + 2ab + b^2} - \frac{1}{a + b} + a$

(6) $\frac{5}{b} + \frac{6}{b^2 + b} - \frac{4}{b^2 - b}$

(7) $\frac{4}{a^2 - x^2} + \frac{5}{a + x} - \frac{2}{a - x}$

(8) $\dfrac{9}{m^2n} - \dfrac{2}{m^2n - m^2} + \dfrac{4}{mn + m}$

(9) $\dfrac{6}{b + x} - 8 + \dfrac{9}{b - x}$

(10) $\dfrac{7}{x^2 + x} + \dfrac{3}{x^2 - x} + \dfrac{5}{x}$

(11) $\dfrac{7}{1 - k} - \dfrac{9}{1 + k} + \dfrac{6}{k}$

(12) $\dfrac{5}{c} + \dfrac{3}{c^2 + c} - \dfrac{8}{c^2 - c}$

(13) $\dfrac{6t^2}{t^2 + 2tu + u^2} - \dfrac{3t}{t + u} + 4$

(14) $\dfrac{9}{a} + \dfrac{3}{a + x} - \dfrac{6}{a - x}$

(15) $\dfrac{2}{3 - 3x} + \dfrac{4}{3 + 3x} - \dfrac{1}{3}$

(16) $\dfrac{10}{8t - 2u} - \dfrac{3}{4t - u} + \dfrac{2}{4t + u}$

(17) $\dfrac{6}{a} + \dfrac{5}{a + x} - \dfrac{12}{a - x}$

(18) $\dfrac{1}{y} - \dfrac{6}{y^2 - 1} - \dfrac{4}{y + 1}$

(19) $\dfrac{12}{c^2} - \dfrac{2}{c^2 - c} + \dfrac{4}{c^2 + c}$

(20) $\dfrac{x}{1 - a} - \dfrac{x}{1 + a} + \dfrac{6}{a}$

(21) $\dfrac{3}{a^2 - 2ab + b^2} - \dfrac{2}{a - b}$

(22) $\dfrac{x - y}{x^2 + 2xy + y^2} - \dfrac{x - y}{x} + \dfrac{1}{y}$

(23) $\dfrac{6}{x^2 - 2xy + y^2} - \dfrac{4}{x^2 - y^2} - \dfrac{3}{x + y}$

(24) $\dfrac{7a + b}{a^2 - b^2} - \dfrac{4a - 3b}{a^2 + 2ab + b^2}$

(25) $\frac{1}{a+x} - \frac{1}{a-x} - \frac{2a-3x}{a^2+2ax+x^2}$

(26) $\frac{1}{x} - \frac{x}{x^2-2xy+y^2} + \frac{2}{x-y}$

(27) $\frac{6c-7d}{c} - \frac{4c+8d}{d} - \frac{9c-d}{cd}$

(28) $\frac{3a+2}{x^2+4x+4} - \frac{a-6}{x+2} + 7$

(29) $\frac{1}{m^2-2mn+n^2} + \frac{2}{m^2+2mn+n^2}$

(30) $\frac{x+y}{x-y} + \frac{2x}{x+y} - \frac{3y^2}{x^2-2xy+y^2}$

VIII. Signs of Fractions

1. If we try to reduce $\frac{a-b}{b^2-a^2}$ to lowest terms we find some difficulty because the order of the letters in the denominator is different from that of the numerator.

$$\frac{a-b}{b^2-a^2} = \frac{a-b}{(b+a)(b-a)}$$

(a) In order to reduce the fractions, we need to change the factor $(b-a)$ to $(a-b)$; but we have no right to do it, unless we make another change also. To find what this change is, let us study the signs of a simpler fraction, $\frac{6}{2}$.

(b) We know $\frac{6}{2}$ is a positive fraction, even though the plus sign is not written before the line separating its terms. Negative $\frac{6}{2}$ is written $-\frac{6}{2}$.

(c) Every fraction has a *sign of the fraction* as a *whole* written, or understood, before the fraction line.

Each term of a fraction has its own sign also. Therefore a fraction has three signs:

(1) the sign of the fraction,
(2) the sign of the numerator,
(3) the sign of the denominator.

(d) Since a fraction is an indicated division, the laws of signs in divisions apply to fractions.

$+\frac{+6}{+2} = +3$	$-\frac{-6}{+2} = -(-3) = +3$
$+\frac{-6}{-2} = +3$	$-\frac{+6}{-2} = -(-3) = +3$

By the equality axiom, therefore,

$$+\frac{+6}{+2} = +\frac{-6}{-2} = -\frac{-6}{+2} = -\frac{+6}{-2};$$

or, with general or literal numbers,

$$+\frac{+a}{+b} = +\frac{-a}{-b} = -\frac{-a}{+b} = -\frac{+a}{-b}.$$

2. (a) How many of the three signs of a fraction may be changed without altering the value?

(b) State three ways in which the signs of a fraction may be changed.

3. (a) Let us apply this law of signs in reducing $\frac{a-b}{b^2-a^2}$ to lowest terms.

(b) This fraction, as every other one, may be written in four possible ways, thus:

$$+\frac{+(a-b)}{+(b^2-a^2)} = +\frac{-(a-b)}{-(b^2-a^2)} = -\frac{-(a-b)}{+(b^2-a^2)} = -\frac{+(a-b)}{-(b^2-a^2)}.$$

(c) The first or original form, we found, could not be reduced.

(d) Let us try the second. By removing the parentheses, $+\dfrac{-(a-b)}{-(b^2-a^2)}$ becomes $+\dfrac{-a+b}{-b^2+a^2}$

or, by reversing the order in each term, $+\dfrac{b-a}{a^2-b^2}$.

Why is this form no better than the first?

(e) By removing the parentheses in the third form,

$-\dfrac{-(a-b)}{+(b^2-a^2)}$, we get $-\dfrac{-a+b}{b^2-a^2}$ which may be written $-\dfrac{b-a}{b^2-a^2}$.

(f) Factor and reduce to lowest terms:

$$-\frac{b-a}{b^2-a^2}=-\frac{\cancel{b-a}}{(b+a)\,(\cancel{b-a})}=-\frac{1}{b+a}=-\frac{1}{a+b}.$$

Why is no change of signs necessary in the last step?

(g) While this form may be used, we prefer the fourth one; for, when possible, we like to keep the letters in an expression in alphabetical order.

Reduce the fractions to lowest terms by using the fourth form.

(h) The fraction may be reduced in its original form;

$$\text{as, }\frac{a-b}{b^2-a^2}=\frac{\overset{-1}{\cancel{a-b}}}{(b+a)\,(\cancel{b-a})}=\frac{-1}{b+a}=\frac{-1}{a+b},$$

but errors are more likely to occur.

IX. Practice Exercises

1. *In Changing Signs*

(a) By changing the necessary signs write each of the following fractions in four forms.

(1) $-\dfrac{x-y}{y^2-x^2}$ (2) $\dfrac{a+c}{c^2-a^2}$ (3) $-\dfrac{1}{-a}$

(4) $-\frac{-2}{3}$ (5) $-\frac{c-d}{d^2-c^2}$ (6) $+\frac{s-t}{t^2-s^2}$

(7) $-\frac{y-x}{x^2-y^2}$ (8) $+\frac{2-x}{x^2-4}$ (9) $-\frac{a+3}{9-a^2}$

(10) $\frac{-x}{y-x}$ (11) $-\frac{-a+b}{b^2-a^2}$ (12) $+\frac{2x-y}{y^2-4x^2}$

2. *In Reducing to Lowest Terms*

(a) Reduce the following fractions to lowest terms and check:

(1) $-\frac{a-d}{d^2-a^2}$ (6) $-\frac{7-x}{x^2-49}$ (11) $\frac{x^2-2xy+y^2}{y-x}$

(2) $\frac{y-x}{x^2-y^2}$ (7) $\frac{r-s}{s^2-r^2}$ (12) $-\frac{16-x^2}{x-4}$

(3) $-\frac{3-b}{b^2-9}$ (8) $\frac{x-c}{c^2-x^2}$ (13) $-\frac{a-5}{25-a^2}$

(4) $-\frac{b-a}{a^2-2ab+b^2}$ (9) $-\frac{l^2-k^2}{k-l}$ (14) $-\frac{-2-a}{a^2+4a+4}$

(5) $-\frac{-c-d}{c^2+2cd+d^2}$ (10) $\frac{4-a^2}{a+2}$ (15) $-\frac{x^2-2ax+a^2}{a-x}$

3. *In Addition and Subtraction*

(a) Simplify the following.

(b) Check:

(1) $\frac{1}{2+x}-\frac{3}{x-2}$ (4) $\frac{4}{c+d}-\frac{5}{d-c}$

(2) $-\frac{a+b}{b^2-a^2}+\frac{1}{a-b}$ (5) $\frac{y-x}{x^2-y^2}-\frac{1}{x+y}-\frac{3}{y-x}$

(3) $\frac{a}{a-b}+\frac{b}{b-a}$ (6) $\frac{2}{a-d}-\frac{5}{d+a}-\frac{3}{d-a}$

(7) $\dfrac{x}{x-y}+\dfrac{y}{y-x}$

(8) $\dfrac{a+b}{x-y}-\dfrac{b-a}{y-x}$

(9) $\dfrac{6}{b-a}-\dfrac{5}{a-b}$

(10) $\dfrac{9}{x^2-y^2}-\dfrac{2}{y-x}$

(11) $\dfrac{x}{a-6}-\dfrac{ax+2x}{36-a^2}$

(12) $\dfrac{12}{x-7}+\dfrac{4x+20}{49-x^2}$

(13) $\dfrac{a}{a-b}-\dfrac{b}{b-a}$

(14) $\dfrac{6}{b^2-a^2}-\dfrac{a+b}{a-b}$

(15) $\dfrac{x-y}{y^2-x^2}+\dfrac{y-x}{x-y}$

(16) $\dfrac{a^2}{a^2-c^2}-\dfrac{a}{c-a}$

(17) $\dfrac{a-t}{a^2-t^2}-\dfrac{6}{t-a}$

(18) $\dfrac{a}{x-y}-\dfrac{b}{y-x}$

(19) $\dfrac{a+b}{a-b}-\dfrac{b-a}{a+b}$

(20) $\dfrac{c-d}{c+d}+\dfrac{4}{d^2-c^2}$

CHAPTER SEVEN

MULTIPLICATION AND DIVISION OF FRACTIONS

A. MULTIPLICATION OF FRACTIONS

I. Fractions with Monomial Denominators

1. Algebraic fractions may be multiplied in the same way as arithmetic fractions.

(a) $6 \times \frac{4}{5} = \frac{24}{5}$

(b) $a \times \frac{b}{c} = \frac{ab}{c}$

(c) $\frac{3}{5} \times \frac{7}{8} = \frac{3 \cdot 7}{5 \cdot 8} = \frac{21}{40}$

(d) $\frac{a}{x} \times \frac{b}{y} = \frac{ab}{xy}$

(e) $\frac{\cancel{2}}{\cancel{3}} \times \frac{\cancel{3}}{4} \times \frac{\cancel{5}}{\underset{3}{\cancel{6}}} \times \frac{\overset{3}{\cancel{9}}}{\underset{2}{\cancel{10}}} = \frac{3}{8}$

(f) $\frac{a}{x} \times \frac{x}{y} \times \frac{b}{ax} \times \frac{x}{2b} = \frac{\cancel{a}\cancel{b}\overset{x^2}{\cancel{x^3}}}{2\cancel{a}\cancel{b}\cancel{x}y} = \frac{x^2}{2y}$

2. Simplify $\frac{3a^2b}{xy^2} \times \frac{2x^3}{4ab} \times \frac{5y^2}{a^2} \times 2 \times \frac{a^3}{12ay}$

(a) The multiplication may be indicated and the like factors in the numerator and denominator canceled, thus:

$$\frac{3a^2b}{xy^2} \times \frac{2x^3}{4ab} \times \frac{5y^2}{a^2} \times 2 \times \frac{a^3}{12ay} = \frac{\cancel{3}\cdot\cancel{2}\cdot 5\cdot\cancel{2}\cdot \cancel{a^2}\cancel{b}\overset{x^2}{\cancel{x^3}}\cancel{y^2}\overset{a}{\cancel{a^3}}}{\cancel{4}\cdot\underset{4}{\cancel{12}}\cdot \cancel{x}\cancel{y^2}\cancel{a}\cancel{b}\cancel{a^2}\cancel{a}y}$$

$$= \frac{5ax^2}{4y}$$

(b) Or, the cancellation may be done first, then the uncanceled factors multiplied.

$$\frac{\cancel{3}\cancel{a^2}\cancel{b}}{\cancel{x}\cancel{y^2}} \times \frac{\overset{x^2}{\cancel{2x^3}}}{\cancel{4ab}} \times \frac{5\cancel{y^2}}{\cancel{a^2}} \times \cancel{2} \times \frac{\overset{a}{\cancel{a^3}}}{\underset{4}{\cancel{12}}\cancel{x}y} = \frac{5ax^2}{4y}$$

II. Fractions with Polynomial Denominators

1. Multiply $\dfrac{2a^2 - ab}{a^2 - 3ab + 2b^2} \cdot \dfrac{a^2 + ab - 2b^2}{4a^2 - 4ab + b^2} \cdot \dfrac{a^2 - 3ab + 2b^2}{a^3 + 2a^2b}$

(a) Factor each term and then cancel the common factors, thus:

$$\frac{\cancel{a}(\cancel{2a-b})}{(\cancel{a-2b})(\cancel{a-b})} \cdot \frac{(\cancel{a+2b})(\cancel{a-b})}{(\cancel{2a-b})(2a-b)} \cdot \frac{(\cancel{a-2b})(a-b)}{\underset{a}{\cancel{a^2}}(\cancel{a+2b})} = \frac{a-b}{a(2a-b)}$$

(b) In this product, why can we not cancel the a of the denominator into the a of the numerator?

(c) In simplifying fractions, always cancel the *whole* of a factor, never one *term* of it.

III. Practice Exercises in Multiplication of Fractions

1. $\dfrac{10a^2x}{by} \times \dfrac{b^2}{3ax^5} \times \dfrac{9y}{5ax^2}$

2. $-\dfrac{18x}{y} \times \dfrac{5y}{6x^2} \times \dfrac{x^3}{10y^2}$

3. $\dfrac{5x^2y}{ab^2} \times \dfrac{2a^3}{6xy} \times \dfrac{3b^2}{x^2} \times 4 \times \dfrac{x^3}{16bx}$

4. $-\dfrac{6ab}{7x} \times \dfrac{-x^2}{b^3y^2} \times \dfrac{14y}{3a^2b} \times \dfrac{-xy}{2}$

5. $\dfrac{a-b}{a+b} \cdot \dfrac{a^2 - b^2}{a^2 - 2ab + b^2} \cdot \dfrac{a^2 + 2ab + b^2}{a+b}$

6. $\dfrac{a^2 - ab - 6b^2}{a^2 - 4ab} \cdot \dfrac{a^3 - 2a^2b}{ab + 5b} \cdot \dfrac{b^3}{a^2 - 5ab + 6b^2}$

7. $\dfrac{x^2 - 2x - 35}{x+3} \cdot \dfrac{x^2 - 1}{x^2 + 3x - 10} \cdot \dfrac{x^2 + x - 6}{x - 7}$

8. $\dfrac{x^2 + 8x + 16}{x^2 + 4x + 4} \cdot \dfrac{x^2 - 8x + 16}{x^2 - 4x + 4} \cdot \dfrac{x^2 - 4}{x^2 - 16}$

9. $\dfrac{a^2 + ax - 2x^2}{a^2 + 2ax} \cdot \dfrac{9a^2 + 6ax + x^2}{a^2 - 4ax + 3x^2} \cdot \dfrac{3a^2 + 15ax}{3a^2 + 16ax + 5x^2}$

10. $\dfrac{b^2 - 2b - 8}{b^2 - 8b + 15} \cdot \dfrac{b^2 - 8b + 12}{b^2 + 9b + 14} \cdot \dfrac{b^2 + 2b - 35}{b^2 - 10b + 24}$

11. $\dfrac{6c^2 - 5cx - 4x^2}{c^2 - x^2} \cdot \dfrac{5c^2 + 4cx - x^2}{8c^3 + 4c^2x} \cdot \dfrac{2c^2 - 2cx}{15c^2 - 23cx + 4x^2}$

12. $\dfrac{2x^2 - 11x + 15}{15x^2} \cdot \dfrac{20x^3 + 65x^2 - 60x}{3x^2 - 4x - 15} \cdot \dfrac{12x^2 + 11x - 15}{2x^2 + 3x - 20}$

B. DIVISION OF FRACTIONS

I. Fractions with Monomial Denominators

1. One who knows how to divide arithmetic fractions will find nothing new or difficult in division of algebraic fractions.

(a) $4 \div \frac{2}{3} = \overset{2}{\cancel{4}} \times \frac{3}{\cancel{2}} = 6$

(b) $a^2 \div \frac{a}{b} = \overset{a}{\cancel{a^2}} \times \frac{b}{\cancel{a}} = ab$

(c) $\frac{2}{3} \div 4 = \frac{\cancel{2}}{3} \times \frac{1}{\underset{2}{\cancel{4}}} = \frac{1}{6}$

(d) $\frac{a}{b} \div a^2 = \frac{\cancel{a}}{b} \times \frac{1}{\underset{a}{\cancel{a^2}}} = \frac{1}{ab}$

(e) $\frac{5}{6} \div \frac{2}{3} = \frac{5}{\underset{2}{\cancel{6}}} \times \frac{\cancel{3}}{2} = \frac{5}{4}$

(f) $\frac{5a}{6b} \div \frac{2c}{3d} = \frac{5a}{\underset{2}{\cancel{6}}b} \times \frac{\cancel{3}d}{2c} = \frac{5ad}{4bc}$

2. How may a problem in division of fractions be changed to one in multiplication?

II. Fractions with Polynomial Denominators

1. If the terms of the fractions are polynomials, factor each first, then multiply the dividend by the inverted divisor. It is necessary to change all integers and mixed expressions to fraction form first.

2. Simplify $\left(x + 1 - \frac{6}{x}\right) \div \left(x + 7 + \frac{12}{x}\right)$.

(a) *Solution:*

$$\left(x + 1 - \frac{6}{x}\right) \div \left(x + 7 + \frac{12}{x}\right) =$$

$$\frac{x^2 + x - 6}{x} \div \frac{x^2 + 7 + 12}{x} = \frac{(x+3)\ (x-2)}{x} \div \frac{(x+3)\ (x+4)}{x}$$

$$= \frac{\cancel{(x+3)}\ (x-2)}{\cancel{x}} \times \frac{\cancel{x}}{\cancel{(x+3)}\ (x+4)}$$

$$= \frac{x - 2}{x + 4}.$$

(b) Check with $x = 3$.

$$\left(x + 1 - \frac{6}{x}\right) \div \left(x + 7 + \frac{12}{x}\right) = \frac{x - 2}{x + 4}$$

$$\left(3 + 1 - \frac{6}{3}\right) \div \left(3 + 7 + \frac{12}{3}\right) = \frac{3 - 2}{3 + 4}$$

$$2 \div 14 \qquad = \frac{1}{7}$$

$$\frac{1}{7} = \frac{1}{7}$$

III. Practice Exercises in Division of Fractions

1. $6 \div \frac{3}{4}$
2. $\frac{5}{8} \div 10$
3. $x^2 \div \frac{x}{a}$
4. $\frac{x}{y} \div x^2$
5. $\frac{3a}{5c} \div \frac{9b}{10d}$
6. $\frac{4a^3t}{3b} \div \frac{6at^2}{15b^2}$
7. $\frac{5ax^2}{2by} \div \frac{15x}{8b}$
8. $\frac{a^2b^3c^5}{x^3yz^4} \div \frac{ab^2c^4}{x^4y^2z^5}$
9. $\frac{a^2 + ab}{ab - b^2} \div \frac{a + b}{a - b}$
10. $\frac{c^2 + cd}{ax + xy} \div \frac{c^3 + c^2d}{ax^3 + x^3y}$

11. $\dfrac{(a+x)(a-x)}{(c+d)(c+d)} \div \dfrac{a-x}{c+d}$

12. $\dfrac{x^2-y^2}{a^2+2ad+d^2} \div \dfrac{x^2+2xy+y^2}{a^2-d^2}$

13. $\dfrac{x^2-5x-14}{x^2+10x+25} \div \dfrac{x-7}{x+5}$

14. $\dfrac{c^2-6c-27}{d^2+2d-8} \div \dfrac{bc-9b}{ad-4a}$

15. $\left(x+4-\dfrac{5}{x}\right) \div \left(x+7+\dfrac{10}{x}\right)$

16. $\left(x+1-\dfrac{6}{x}\right) \div \left(x-3-\dfrac{18}{x}\right)$

17. $\left(2x-13+\dfrac{15}{x}\right) \div \left(4x-27+\dfrac{35}{x}\right)$

18. $\left(15a+14-\dfrac{8}{a}\right) \div \left(10a-19+\dfrac{6}{a}\right)$

19. $\left(8y-10-\dfrac{7}{y}\right) \div \left(12y-5-\dfrac{28}{y}\right)$

20. $\left(b+4-\dfrac{5}{b}\right) \div \left(b+9+\dfrac{20}{b}\right)$

IV. Complex Fractions

1. (a) By checking prove that $\frac{5}{6}$ is a root of the following equation

$$\frac{x+1}{3x+1} = \frac{11}{21}.$$

(b) In checking $x = \dfrac{5}{6}$, the equation becomes

$$\frac{\dfrac{5}{6}+1}{\dfrac{5}{2}+1} = \frac{11}{21}.$$

In this form the first member is a *complex fraction*, that is, a fraction containing one or more fractions either in the numerator, or in the denominator, or in both.

(c) Name the fraction in the numerator of the complex fraction; in its denominator.

2. To change the complex fraction to a simple fraction, we must get rid of the denominators of the fractions in each of its terms. This may be done by multiplying each term of the complex fraction by the L. C. D. of $\frac{5}{6}$ and $\frac{5}{2}$, which is 6, as:

$$\frac{\frac{5}{6}+1}{\frac{5}{2}+1} \times \frac{6}{6} = \frac{5+6}{15+6} = \frac{11}{21}.$$

(a) Why have we the right to multiply thus?

3. Simplify:

(a) $\dfrac{4+\frac{1}{5}}{1-\frac{2}{5}}$ (b) $\dfrac{\frac{1}{2}+\frac{1}{3}}{\frac{2}{3}+\frac{1}{4}}$ (c) $\dfrac{\frac{1}{5}+\frac{1}{6}}{\frac{1}{2}+\frac{1}{3}}$

(d) $\dfrac{\frac{8}{9}-2}{3-\frac{5}{9}}$ (e) $\dfrac{\frac{1}{2}+\frac{1}{3}+\frac{1}{4}}{\frac{2}{3}+\frac{3}{4}+\frac{4}{5}}$ (f) $\dfrac{\frac{3}{4}}{5}+\dfrac{3}{\frac{4}{5}}$

4. Complex fractions with literal terms may be simplified in the same way. Be sure to multiply each term in both the numerator and the denominator by the lowest common denominator (L. C. D.) of the fractional terms.

5. (a) Simplify $\dfrac{\frac{a}{x}+1}{\frac{a}{y}+1}$

(b) Multiplying both terms by xy, we have

$$\frac{ay+xy}{ax+xy}, \text{ which factors into } \frac{y(a+x)}{x(a+y)},$$

6. (a) Simplify $\dfrac{\dfrac{1}{x} + \dfrac{1}{y}}{\dfrac{1}{x^2} - \dfrac{1}{y^2}}$

(b) Multiplying both terms by x^2y^2, we have $\dfrac{xy^2 + x^2y}{y^2 - x^2}$,

which factors into $\dfrac{xy(y + x)}{(y + x)\,(y - x)}$

and reduces to $\dfrac{xy}{y - x}$

7. Simplify:

(a) $\dfrac{2x - 3 + \dfrac{1}{x}}{2 - \dfrac{1}{x}}$ (b) $\dfrac{x - \dfrac{1}{x}}{\dfrac{1}{x} - \dfrac{1}{x^2}}$

(c) $\dfrac{1 + \dfrac{1}{a - 1}}{1 - \dfrac{1}{a + 1}}$ (d) $\dfrac{\dfrac{1}{a} + \dfrac{1}{b} + \dfrac{1}{c}}{\dfrac{1}{abc}}$

(e) $\dfrac{\dfrac{1}{a} + \dfrac{1}{b} + \dfrac{1}{c}}{\dfrac{a}{b} + \dfrac{b}{c} + \dfrac{c}{a}}$ (f) $\dfrac{\dfrac{2}{x} + \dfrac{2}{y} + \dfrac{2}{z}}{\dfrac{x}{y} + \dfrac{y}{z} + \dfrac{z}{x}}$

(g) $\dfrac{3a - 5 + \dfrac{2}{a}}{3 - \dfrac{2}{a}}$ (h) $\dfrac{4b + 7 + \dfrac{3}{b}}{4 + \dfrac{3}{b}}$

(i) $\dfrac{3 + \dfrac{3}{x - 1}}{3 - \dfrac{3}{x + 1}}$ (j) $\dfrac{\dfrac{4}{x} + \dfrac{4}{y} + \dfrac{4}{z}}{\dfrac{4}{xyz}}$

CHAPTER EIGHT

SIMPLE EQUATIONS WITH TWO UNKNOWNS

A. MEANING OF A SYSTEM OF EQUATIONS

1. If you are told that a boy bought 5 pencils and 4 tablets for 85 cents, you do not know the price of the pencils nor of the tablets. But if you know that another boy bought 3 pencils and 4 tablets of the same kind for 75 cents, you can calculate the price of each article.

(a) The statements of these two transactions put into equation form are:

$$5p + 4t = 85 \qquad (1)$$
$$3p + 4t = 75 \qquad (2)$$

(b) Subtracting the second equation from the first gives $2p = 10$, which means 2 pencils cost 10 cents. Therefore one pencil costs 5 cents; 3 pencils cost 15 cents, and 5 pencils cost 25 cents.

(c) By substituting 25 for $5p$ in the first equation, we get $25 + 4t = 85$.

Solving this equation gives

$$4t = 60$$
$$t = 15$$

Therefore, each tablet costs 15 cents.

(d) Check these prices by substituting them for the letters in the second equation, thus:

$$3p + 4t = 75$$
$$15 + 60 = 75$$
$$75 = 75$$

2. When we knew of only one transaction with two articles whose prices were unknown, we had no means of finding the prices. But with two transactions, we could find the two unknowns. That is, *two equations are required to solve a problem which has two unknown quantities.*

3. (a) Suppose the first boy bought 5-cent pencils and 15-cent tablets, but the second boy bought penny pencils and 10-cent tablets.

The two equations then would be

$$5p + 4t = 85$$
$$3p + 4t = 43$$

(b) Why would this set of equations not help us to find either price?

(c) In both equations p stands for the price of a pencil, but not the same price; t stands for the price of a tablet, but for a different price in each equation.

4. (a) If one boy bought 5 pencils and 4 tablets for 85 cents and another boy bought 10 pencils and 8 tablets for \$1.70 or 170 cents, could you calculate the cost of each?

(b) These transactions give the equations,

$$5p + 4t = 85$$
$$10p + 8t = 170$$

in which the second equation is just twice the first.

(c) These two equations tell practically the same story, that is, they show the same *relation between the two variables*, p and t.

5. (a) We see, therefore, that to find the values of two unknown quantities, we must have a *set* of *two equations* with the *same two variables*, but each equation must show *different relations between* the variables.

(b) Such a set of equations is called a *system* of *simultaneous equations.*

(c) The word *simultaneous* comes from a Latin word which means *at the same time.* What must be true *at the same time,* if a set of equations may be called simultaneous?

6. (a) The first two equations about the pencils and tablets are simultaneous.

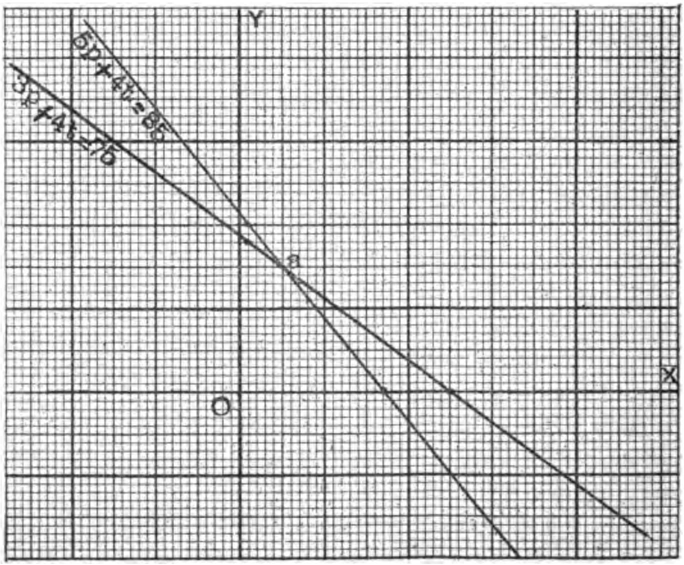
$$5p + 4t = 85$$
$$3p + 4t = 75$$

(b) What is the degree of each of these equations?

(c) These equations form a *system* of *simultaneous linear equations,* which, for brevity's sake, may be called a *system* of *linear equations.*

B. GRAPH OF A SYSTEM OF LINEAR EQUATIONS

1. (a) On the same pair of axes, draw a graph of each of these equations.



(b) Why is the graph of each equation a straight line?

(c) In how many points can two straight lines intersect?

(d) Locate the point of intersection, a, of these two graphs.

(e) Point $a = (5, 15)$, that is, the x-distance is 5 and the y-distance is 15.

(f) In solving these two equations the value of p was found to be 5 and of t, 15. How do these values of the two unknowns compare with the x-distance and y-distance of the intersection point?

2. (a) A set of simultaneous equations may be *solved graphically* by locating the point of intersection of the graphs of the two equations.

(b) The x-distance of the intersection point is the value of the first variable and the y-distance is the value of the second variable.

3. (a) The sum of two numbers is 15. Their difference is 3. Find the numbers.

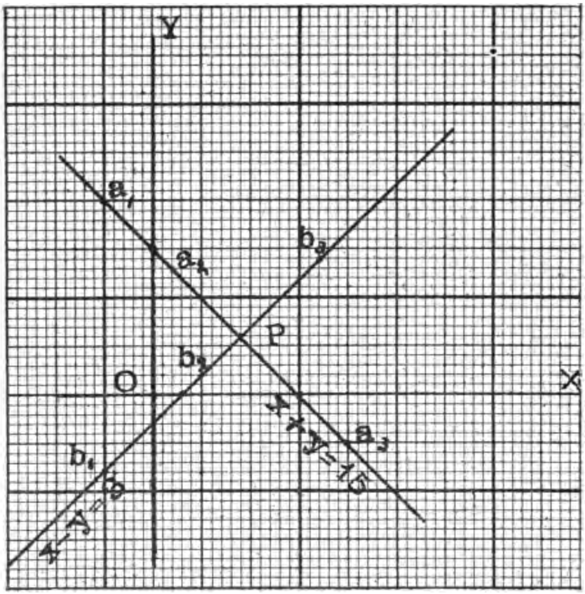
$$\begin{aligned} x + y &= 15 \\ x - y &= 3 \end{aligned}$$

(b) Draw the graphs of the two equations.

(c) In plotting the second equation, by letting $y = 0$, we get $x = 3$. Therefore, one of the plotting points is $(3, 0)$. By letting $x = 0$, we get $y = -3$. Therefore, the second plotting point is $(0, -3)$; that is, it is on the y-axis, 3 units below the origin.

(d) The location of P, the intersection point, is $(9, 6)$. Therefore, $x = 9$ and $y = 6$.

(e) Prove by substituting these values in both of the original equations and then compare with the wording of the problem.



(1) Why is it necessary to check in both ways?

$x + y = 15$	$x - y = 3$
$9 + 6 = 15$	$9 - 6 = 3$
$15 = 15$	$3 = 3$

(2) The sum of 9 and 6 is 15. The difference between 9 and 6 is 3.

4. (a) The squared paper used for graphs is sometimes called *coördinate* paper. The *x-distance* and the *y-distance* of a point are called its *coördinates.* In the preceding graph, the coördinates of P are (9, 6). It is sometimes more convenient to use one word to refer to both of the distances.

(b) Find the coördinates of points a_1, a_2, and a_3. Do these coördinates satisfy the equation $x + y = 15$? Do they satisfy $x - y = 3$?

(c) Find the coördinates of points b_1, b_2, and b_3. Which equation do these coördinates satisfy?

5. Show that a system of simultaneous linear equations can have but one set of roots.

6. (a) Solve the following systems of linear equations by drawing their graphs.

(b) Check the roots in both equations.

(1)	$x + y = 12$ $x - y = 4$	(6)	$x - y = -6$ $x + 3y = 10$
(2)	$2x + y = 8$ $x - y = 1$	(7)	$x + y = -8$ $2x + y = -11$
(3)	$7a - 2b = 22$ $5a + 2b = 26$	(8)	$4x + y = 5$ $x - y = 5$
(4)	$5c - 2n = 30$ $3c + 4n = 44$	(9)	$2c + d = 18$ $3c - d = 7$
(5)	$3p + 2t = 35$ $2p + 3t = 40$	(10)	$2t - 3u = 17$ $3t - u = 1$

I. Historical Note

The making of geometric pictures of algebraic equations is an invention of a Frenchman, René Descartes (dā kärt′), in 1638. Descartes was born near Tours in 1596. As a lad, on account of ill health, he developed the habit of lying in bed in the mornings. In later life he clung to this habit on the plea that only in this way could he do good work in mathematics and preserve his health.

Following the custom of a man in his position, he joined the army at twenty-one. He claims that his first ideas

about coördinates came from three dreams he had one November night in 1619, when he was campaigning on the Danube. He was the first to introduce our modern method of writing exponents. He fixed the custom of using the first letters of the alphabet to denote known quantities and the last letters of the alphabet to denote unknown quantities.

In order to be free to study and write, he moved to Holland, where he lived twenty years. He wrote on many subjects other than mathematics, including physics, optics, astronomy, and philosophy.

In 1649, he went to Stockholm on the invitation of the Queen of Sweden. He died there a few months after his arrival.

C. ALGEBRAIC SOLUTIONS OF SYSTEMS OF LINEAR EQUATIONS

I. Elimination by Addition or Subtraction

Sometimes it is difficult to read the coördinates of the intersection of the graphs of two simultaneous equations. In such cases it is better to use an algebraic method of solution instead of the graphic.

1. In the discussion of the problem about pencils and tablets (p. 122), we used one algebraic method when we subtracted one equation from the other, in order to get rid of one of the unknown quantities.

2. (a) The process of getting rid of one of the variables in a system by adding or subtracting the equations is called *elimination* by *addition* or *subtraction*.

$$\begin{array}{rcll} 5p + 4t & = & 85 & \quad (1) \\ 3p + 4t & = & 75 & \quad (2) \\ \hline 2p & = & 10 & \quad (3) \\ p & = & 5 & \quad (4) \end{array}$$

(b) To find the value of the second variable, substitute the value of the first in either equation and solve, as:

$$\begin{aligned} 5p + 4t &= 85 \\ 25 + 4t &= 85 \\ 4t &= 60 \\ t &= 15 \end{aligned}$$

3. (a) In this set of equations, the coefficients of one variable, t, are the same in both equations. Therefore, they are ready for the subtraction immediately. If, however, the coefficients of the variable to be eliminated are unlike in the two equations, it is necessary to make them alike. This may be done by multiplying one of the equations by a convenient number. If necessary, both equations may be so multiplied.

(b) To solve the equations,

$$5x + 3y = 34 \qquad (1)$$
$$3x - 4y = 3 \qquad (2)$$

decide first which variable you desire to eliminate and then multiply the equations by the numbers necessary to make the coefficients of that variable alike.

(c) To eliminate y, multiply the first equation by 4 and the second by 3, giving

$4 \times (1)$,	$20x + 12y = 136$	(3)
$3 \times (2)$,	$9x - 12y = 9$	(4)
By addition.	$29x = 145$	(5)
By division,	$x = 5$	(6)
Substitution in (1),	$25 + 3y = 34$	(7)
	$3y = 9$	(8)
	$y = 3$	(9)

$$\therefore x = 5$$
$$y = 3$$

(d) *Check.*

$$\begin{array}{rcl|rcl} 5x + 3y &=& 34 & 3x - 4y &=& 3 \\ 25 + 9 &=& 34 & 15 - 12 &=& 3 \\ 34 &=& 34 & 3 &=& 3 \end{array}$$

(e) Why do we eliminate by addition in this system?

(f) To eliminate x, by what number must the first equation be multiplied? the second equation? Solve this system by eliminating x. Must it be done by addition or by subtraction?

4. (a) In the following systems, eliminate one of the variables by addition or subtraction.

(b) *Check the results.*

(1) $\begin{aligned} 5a + 3b &= 42 \\ 3a - 4b &= 2 \end{aligned}$

(2) $\begin{aligned} 3x - 2y &= 17 \\ 4x + y &= 30 \end{aligned}$

(3) $\begin{aligned} 4a + 3x &= 7 \\ 5a + 6x &= 2 \end{aligned}$

(4) $\begin{aligned} 3c - 5d &= -7 \\ 5c - 3d &= 15 \end{aligned}$

(5) $\begin{aligned} 5t + 8u &= 4 \\ 2t + 3u &= 1 \end{aligned}$

(6) $\begin{aligned} 2h - 3k &= 8 \\ 5h - 3k &= -7 \end{aligned}$

(7) $\begin{aligned} x + 5y &= 7 \\ 2x - y &= -19 \end{aligned}$

(8) $\begin{aligned} 4b + 3c &= -3 \\ 5b + 2c &= 5 \end{aligned}$

(9) $\begin{aligned} 2x - 5y &= 6 \\ 3x - 8y &= 8 \end{aligned}$

(10) $\begin{aligned} x + 2y &= 3 \\ 5x + 4y &= -9 \end{aligned}$

II. Elimination by Substitution

Sometimes it is more convenient to solve systems of linear equations by a second method, called *elimination* by *substitution.*

1. (a) The following example illustrates its use.

(b) Solve the system

$$5x - 6y = 32 \qquad (1)$$
$$2x + 5y = -2 \qquad (2)$$

(c) *Solution:*

From (1),

$$x = \frac{32 + 6y}{5} \qquad (3)$$

Substituting $\frac{32 + 6y}{5}$ for x in (2),

$$\frac{64 + 12y}{5} + 5y = -2 \qquad (4)$$

(4) × 5,

$$64 + 12y + 25y = -10 \qquad (5)$$

Collecting terms,

$$37y = -74 \qquad (6)$$

(6) ÷ 37, $\quad y = -2 \qquad (7)$

Substituting -2 for y in (3),

$$x = \frac{32 - 12}{5} = 4 \qquad (8)$$

$$\therefore x = 4$$
$$y = -2$$

(d) Check.

$5x - 6y = 32$	$2x + 5y = -2$
$20 + 12 = 32$	$8 - 10 = -2$
$32 = 32$	$-2 = -2$

2. *To solve a system of equations by the substitution method of elimination:*

(a) Solve either equation for one unknown in terms of the other.

(b) In the other equation, substitute this value of the second variable and solve the equation for the first variable.

(c) Substitute this last value in any preceding equation which contains both variables and solve for the value of the second.

3. The method of elimination by substitution is preferable when a very simple expression can be found which represents one unknown in terms of the other.

4. (a) Solve by the substitution method.
   (b) Check results.

(1) $\begin{aligned} n - 3d &= 3 \\ 2n + 4d &= 16 \end{aligned}$	(5) $\begin{aligned} 4g + 5k &= 6 \\ g - 4k &= 12 \end{aligned}$
(2) $\begin{aligned} 7x - 8y &= 5 \\ 3x + y &= 11 \end{aligned}$	(6) $\begin{aligned} 5t + 4u &= 4 \\ t + u &= 2 \end{aligned}$
(3) $\begin{aligned} 2x + 3y &= 22 \\ x - y &= 1 \end{aligned}$	(7) $\begin{aligned} a - b &= 4 \\ 2a + b &= -22 \end{aligned}$
(4) $\begin{aligned} b + 3x &= 2 \\ 2b + 5x &= 1 \end{aligned}$	(8) $\begin{aligned} x + 6y &= -20 \\ 4x - y &= -5 \end{aligned}$

D. REDUCTION TO STANDARD FORM

Thus far, all of the simultaneous equations have been given in the *standard form*, as $2x + 5y = 16$. Equations containing fractions or parentheses must be simplified to the standard form before a variable can be eliminated.

1. Change to standard form and solve:

   (a) *Solution:*

$$\frac{5}{x+2} - \frac{1}{y-1} = 0 \qquad (1)$$

$$4(x-1) - 3(y-1) = 5 \qquad (2)$$

$(1) \times (x+2)(y-1)$, $\quad 5(y-1) - (x+2) = 0 \quad (3)$

Expanding (3), $\quad 5y - 5 - x - 2 = 0 \quad (4)$

Collecting in (4), $\quad 5y - x = 7 \quad (5)$

Expanding (2), $\quad 4x - 4 - 3y + 3 = 5 \quad (6)$

Collecting in (6), $\quad 4x - 3y = 6 \quad (7)$

Use the equations (5) and (7) as the system and solve by the most convenient method.

2. (a) Reduce the following equations to standard form.
(b) Solve the systems by the most convenient method.
(c) Check results.

(1) $\dfrac{6}{x+2} - \dfrac{1}{y-2} = 0$
$5(x-1) - 4(y-1) = 7$

(2) $\dfrac{8}{a-1} - \dfrac{2}{b-1} = 0$
$7(a-4) - 3(b-1) = 4$

(3) $\dfrac{12}{c+6} - \dfrac{6}{d+1} = 0$
$4(c-2) + 2(d-3) = 20$

(4) $\dfrac{5}{n-2} = \dfrac{8}{d+6}$
$7(n-4) = 3(d+5)$

(5) $2(g-4) + 5(n-1) = -2$
$\dfrac{9}{3g-3} + \dfrac{5}{n+2} = 0$

(6) $3(2r-7) - 4(s+6) = 9$
$\dfrac{3}{r-4} - \dfrac{4}{s+7} = 0$

(7) $\dfrac{5x+12}{2} - \dfrac{4y+7}{3} = 0$
$8(x+3) - 5(y+2) = 3$

(8) $\dfrac{2s-3t}{4} = \dfrac{5s-8t}{9}$
$3(s-t) - 2(2s-3t) = 1$

(9) $\dfrac{5}{a+2} - \dfrac{6}{3-c} = 0$
$4(a-2) - 3(c+6) = -5$

(10) $\dfrac{2x-6}{2} - \dfrac{3y-1}{2} = 0$
$4(x-3) - 3(y+2) = 1$

(11) $\dfrac{5m+3n}{3} - \dfrac{4m-3n}{6} = 0$
$5(m-n) - 4(2m-3n) = 69$

(12) $\dfrac{4(2c-1)}{3} - \dfrac{(5-d)}{2} = 0$
$8(c-d) - 7(4c+d) = 5$

(13) $\dfrac{5}{2x+y} = \dfrac{7}{3x+5}$
$2(x-2) - 3(y+7) = -20$

(14) $\dfrac{7}{5-h} - \dfrac{12}{k+3} = 0$
$6(h+5) - 7(k-7) = 4$

(15) $\dfrac{8}{p+5} - \dfrac{11}{t+4} = 0$
$3(p-1) + 2(t+4) = 28$

(16) $\dfrac{10}{a-4} + \dfrac{9}{b+12} = 0$
$3(a+8) + 2(b-2) = -4$

E. PROBLEMS WITH TWO VARIABLES

I. Suggestions for Solutions

1. In solving problems with two unknowns, represent each variable by its initial letter, when possible.

(a) For example, if a problem is about two numbers, one must be the greater and one the less. Let g equal the greater number and l equal the less.

(b) If the problem is about a fraction, let n equal the numerator and d equal the denominator.

2. Two conditions or statements in the problem must be translated into two equations, in order to form a system.

3. Solve the system by the shortest method.

4. Checking results in the equations proves that you solved the equations correctly, but it does not prove that you made correct equations for the problem. In problems, the only real proof is by checking results in the statements or conditions given in the problem.

II. Practice Exercise in Problems with Two Variables

1. The sum of two numbers is 27. The difference is 3. Find the numbers.

2. Find two numbers whose sum is 11 and whose difference is 5.

3. The value of a fraction is $\frac{3}{4}$. If 4 is subtracted from both numerator and denominator, the value is $\frac{2}{3}$. What is the fraction?

4. If 7 is added to both numerator and denominator of a fraction, its value becomes $\frac{4}{5}$. If 3 is subtracted from both numerator and denominator, its value becomes $\frac{2}{5}$. Find the fraction.

5. Three apples and 2 bananas cost 12 cents. Two apples and 3 bananas cost 13 cents. What was the cost of each?

6. A basket ball team played 40 games. Twice the number lost was 4 less than the number won. How many games were won and how many lost?

7. A boy bought 24 apples and oranges, the apples at $2\frac{1}{2}$ cents each and the oranges at 5 cents each. He paid $1 for all of them. How many of each did he buy?

8. If a number be subtracted from 3 times another number, the difference is 30. The sum of 4 times the first number and 3 times the second is 15. Find the numbers.

9. The perimeter of a rug is 28 feet. The length is 2 feet more than the width. Find the length and width.

10. A boy has a garden in the shape of an isosceles triangle. The total distance around is 80 ft. The sides are 10 ft. longer than the base. How long is each side of the garden?

11. The length of a classroom is 4 ft. more than the width. The total distance around the room is 72 ft. Find the length and the width.

12. A boy has saved $800. He wishes to make $34 a year on it. He can invest part of the money at 4 % and part at 5 %. How much should he put in each investment in order to make $34?

13. If 1 be added to the numerator of a certain fraction, the value of the resulting fraction is $\frac{3}{4}$. If two be added to the denominator, the resulting fraction is $\frac{1}{2}$. Find the fraction.

14. Two girls went to the grocer for sugar and rice. One girl bought 2 pounds of sugar and 3 pounds of rice for 38 cents. The other girl bought 3 pounds of sugar and two pounds of rice for 42 cents. Find the price a pound of each.

15. Two boys were racing to see which one could solve the most extra problems for a lesson. Both together solved 200 problems. One boy lacked 40 of solving twice as many as the other. How many did each solve?

16. A boy tried to see how much money he could save from his earnings. In two weeks he saved 3 dollars. One-fifth of the amount he saved the first week was $1.50 less than the amount he saved the second week. How much did he save each week?

CHAPTER NINE

QUADRATIC EQUATIONS WITH ONE UNKNOWN

A. STANDARD FORM OF A QUADRATIC

1. The standard form of a quadratic equation is $x^2 + ax + b = 0$, in which a represents any number used as a coefficient of x and b represents any number. In the standard form the coefficient of x^2 is 1. The first member of the standard equation is a quadratic trinomial and the second member is 0.

2. The quadratic equation $x^2 + 3x = 10$ changed to standard form is $x^2 + 3x - 10 = 0$. Similarly, $6x^2 - 19x = 7$ becomes $x^2 - \frac{19x}{6} - \frac{7}{6} = 0$.

B. FIRST METHOD OF SOLVING A QUADRATIC— BY FACTORING

1. We have already learned how to solve those quadratic equations in which factoring can be used.

2. Review Chapter Three, Section D.

C. SECOND METHOD — BY GRAPHS

I. Graphs of Quadratic Expressions

1. Many quadratics may be solved by graphs, of a kind very different from those of linear equations. Before we try to graph quadratic equations, let us see what kind of geometric picture a quadratic expression makes.

2. How does a quadratic expression differ from a quadratic equation? $x^2 - 2x - 15$ is a quadratic expression, but $x^2 - 2x - 15 = 0$, is a quadratic equation.

3. (a) Does the expression $x^2 - 2x - 15$ have a fixed numerical value? Why?

(b) What value does it have if $x = 0$?

(c) Find its value when $x = 3$; when $x = -5$.

4. (a) We see, therefore, that the *value* of the *trinomial varies* as the value of x changes. The expression has a different value for each value of x.

(b) Assume a series of different values for x and calculate the value of $x^2 - 2x - 15$ which corresponds with each, as in the table.

x	$x^2 - 2x - 15$
0	-15
1	-16
2	-15
3	-12
4	-7
5	0
6	9
7	20
8	33
-1	-12
-2	-7
-3	0
-4	9
-5	20
-6	33

(c) On a pair of axes, measure the values of x on the x-axis. Plot each point listed in the table, and draw a smooth curve through the points. This curve is the graph of the expression $x^2 - 2x - 15$.

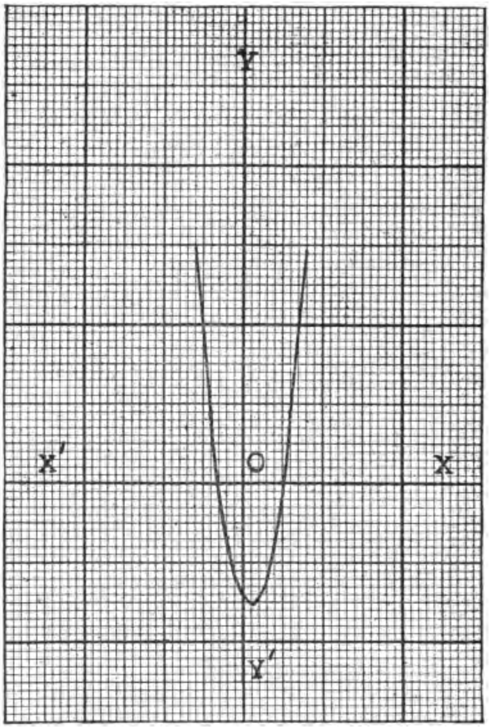
5. (a) From the graph, read the value of $x^2 - 2x - 15$ when x is 5; -4; 7; 0; 6; 2.

(b) We see that the curve cuts the x-axis twice. These intersections show that $x^2 - 2x - 15$ equals *zero* for two different values of x; namely, when $x = 5$ and $x = -3$.

6. (a) In this way make a graph of the expression $x^2 - 4x + 3$.

(b) In how many places does its graph cut the x-axis?

(c) For what values of x does $x^2 - 4x + 3$ equal zero?



GRAPH OF $x^2 - 2x - 15$

7. (a) How does the shape of the graph of a quadratic expression differ from that of a linear expression?

(b) The graphs of *first degree* expressions are *always straight* lines; those of *second degree* or *quadratic* expressions are *always curved lines.*

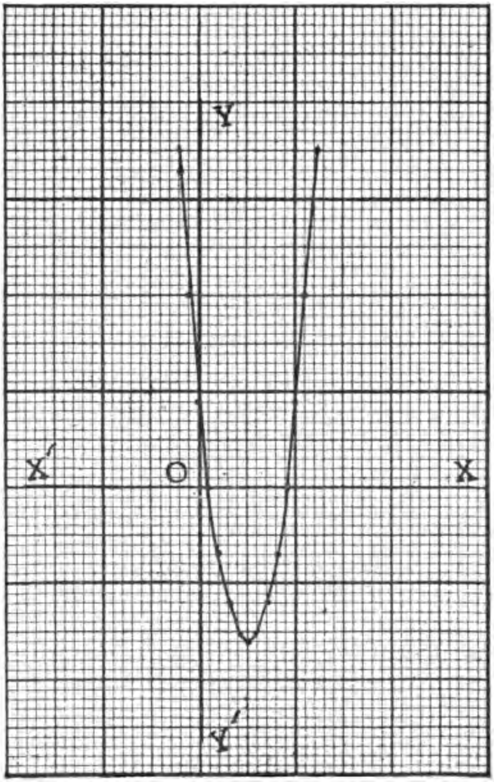
(c) The graphs of such quadratic trinomials have the shape of the letter *U* and are called *parabolas.*

II. Reading a Parabola

1. From the graph of $x^2 - 10x + 9$, find the answers to the following questions:

(a) In how many places does the curve cut the x-axis?

(b) What is the value of x at these points?



GRAPH OF $x^2 - 10x + 9$

(c) For what value, or values, of x is $x^2 - 10x + 9$ equal to 3, -1, 4, 0, 5?

(d) What is the value of the expression at the lowest point on the curve?

(e) What value of x gives the least value to the expression?

III. Graphs of Quadratic Equations

1. By changing a quadratic equation to standard form, we get a *quadratic expression* for the *first member* and *zero* for the *second;* as, $x^2 - 10x + 9 = 0$.

2. (a) To solve this equation *graphically*, draw the graph of the first member.

(b) Since only those values of x which make the expression equal zero can satisfy the equation, note the values of x at the points where the parabola cuts the x-axis. These values are the roots of the equation.

3. (a) How many roots are there in a linear equation?

(b) How many are there in a quadratic equation?

(c) How does the graph show the number of the roots?

4. Solve $x^2 - 10x + 9 = 0$ by factoring and compare the results with the graphic solution.

5. Solve the following uadratic equations by graphs:

(a) $x^2 + 5x + 6 = 0$	(e) $x^2 - 4x - 12 = 0$
(b) $x^2 + 8x + 7 = 0$	(f) $x^2 + 7x + 10 = 0$
(c) $x^2 + 11x + 24 = 0$	(g) $x^2 + 2x - 24 = 0$
(d) $x^2 + x - 12 = 0$	(h) $x^2 - 9x + 18 = 0$

D. PREPARATION FOR THE THIRD METHOD OF SOLVING A QUADRATIC EQUATION—BY COMPLETING THE SQUARE

1. (a) You have found the factoring method very convenient for solving certain quadratic equations, but it can not be used to solve all such equations.

(b) Try solving $x^2 + 4x = 8$ by factoring.

2. (a) The graphic method is tedious and is not always exact. It is interesting to see how the geometric picture of a quadratic differs from that of a linear equation. It helps us to understand their distinction, but, as a method of solution, it is not satisfactory.

3. The next method we study is one that can be used for solving all quadratic equations. It is called the *method* of *completing* the *square*. From the explanation, you will understand why it is so named.

4. Before learning the method, it is necessary to know how to recognize complete or perfect squares. You can recognize the smaller arithmetic numbers that are squares, and you have learned to take the square root of the larger ones.

I. Perfect Squares and Surds

1. (a) Which of the following quantities are perfect squares?

$$4,\quad 6,\quad 12,\quad 16,\quad 24,\quad 25,\quad 125,\quad 196.$$
$$a,\quad ab,\quad a^2b,\quad a^4,\quad a^3b,\quad x^2,\quad x^3,\quad c^2d^2.$$
$$\frac{1}{4},\quad \frac{1}{a^2},\quad \frac{1}{12},\quad \frac{1}{a^2b},\quad \frac{3}{4},\quad \frac{3}{ab},\quad \frac{1}{x^3},\quad \frac{1}{c^2d^2}.$$

(b) Give the square roots of the perfect squares.

2. The square roots of the numbers that are not perfect squares may be expressed by writing the number under the radical sign, as $\sqrt{12}$, $\sqrt{a^2b}$. Such indicated square roots of numbers that are not perfect squares are *quadratic surds*.

3. (a) Some surds may be simplified by taking out the square root of certain factors that are perfect squares and leaving the others under the radical sign; as,

(1) $\sqrt{12} = \sqrt{2^2 \cdot 3}$
$= \pm 2\sqrt{3}$

(2) $\sqrt{a^3b} = \sqrt{a^2 \cdot a \cdot b}$
$= \pm a\sqrt{ab}$

(3) $\sqrt{24} = \sqrt{2^2 \cdot 2 \cdot 3}$

$= \pm 2\sqrt{6}$

(b) You have also learned that no radical is in its simplest form if a fraction is under the radical sign. To get rid of the denominator, it is necessary to make every factor in it a *perfect square,* so that its root may be taken, thus:

(1) $\sqrt{\frac{1}{12}} = \sqrt{\frac{1}{2^2 \cdot 3}}$

$= \sqrt{\frac{1}{2^2 \cdot 3} \times \frac{3}{3}}$

$= \pm \frac{1}{2 \cdot 3}\sqrt{3}$

$= \pm \frac{1}{6}\sqrt{3}$

(2) The approximate decimal value of

$\pm \frac{1}{6}\sqrt{3} = \pm \frac{1}{6} \times 1.73205 +$

$= \pm .28867 +$

(3) $\sqrt{\frac{8x^2y}{3a^3b^2c}} = \sqrt{\frac{2^2 \cdot 2 \cdot x^2y}{3 \cdot a^2 \cdot a \cdot b^2 \cdot c}}$

$= \sqrt{\frac{2^2 \cdot 2 \cdot x^2y}{3 \cdot a^2 \cdot a \cdot b^2 \cdot c} \times \frac{3ac}{3ac}}$

$= \pm \frac{2 \cdot x}{3a \cdot a \cdot b \cdot c}\sqrt{6acy}$

$= \pm \frac{2x}{3a^2bc}\sqrt{6acy}$

4. (a) Simplify the following:

(b) Find the approximate decimal values.

(1) $\sqrt{48}$

(2) $\sqrt{a^5b}$

(3) $\sqrt{72}$

(4) $\sqrt{\frac{1}{6}}$

(5) $\sqrt{\dfrac{12a^2b}{5x^3y^2z}}$

(6) $\sqrt{32}$

(7) $\sqrt{x^6y}$

(8) $\sqrt{\frac{2}{5}}$

(9) $\sqrt{75}$

(10) $\sqrt{288}$

(11) $\sqrt{18a^2b^3c}$

(12) $\sqrt{\dfrac{5}{12}}$

(13) $\sqrt{96}$

(14) $\sqrt{\frac{1}{3}}$

(15) $\sqrt{4a - 4b}$

(16) $\sqrt{g^2k^5}$

(17) $\sqrt{180}$

(18) $\sqrt{\frac{3}{4}}$

II. Completing the Square in Algebraic Polynomials

1. What is the area of a square whose edge is $x + a$? $2a - b$? $3x - 4y$?

2. (a) Which of the following expressions represent areas of squares?

(b) What areas may the others represent?

(c) What are the dimensions of each figure?

(1) $a^2 + 2ab + b^2$

(2) $a^2 - 4ab + 3b^2$

(3) $x^2 - 6x + 9$

(4) $m^2 - 4mn + 4n^2$

(5) $4c^2 - 12cd + 9d^2$

(6) $x^2 + 3x - 10$

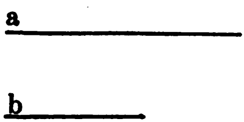
(7) $x^2 - 2ax$

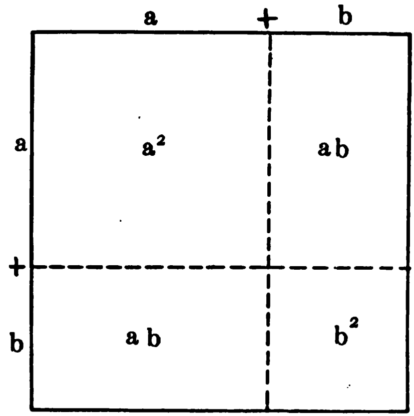
(8) $9x^2 - 6x + 1$

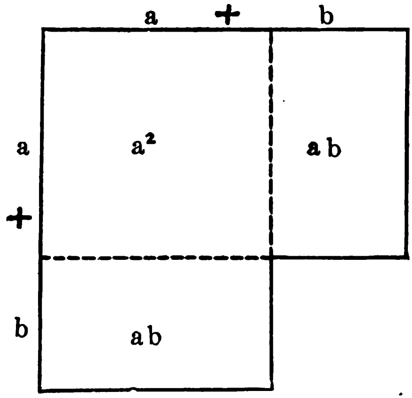
(9) $a^2 + 4a$

(10) $25b^2 - 10bc + c^2$

3. (a) The expression $a^2 + 2ab + b^2$ may be represented as the square of the sum of two lines a and b.







(b) The following figure would have an area of $a^2 + 2ab$.

(c) What does this figure lack to make it a *complete square?*

4. (a) Draw similar figures to represent the following areas.

(b) Find what figure is needed to make a complete square.

(1) $x^2 + 4xy$
(2) $a^2 + 6ab$
(3) $b^2 + 4b$
(4) $c^2 + 8cd$
(5) $a^2 + 12ab$
(6) $a^2 + 10a$

5. (a) If we keep in mind that $(x + y)^2 = x^2 + 2xy + y^2$ and that the middle term of the trinomial is always *twice* the product of the square roots of the first and third terms, it will not be difficult to find the third term, that is, to *complete* the the square.

(b) For example, in $x^2 + 6xy$, we know that x is the root of x^2; that $6xy$ must contain *twice* x or $2x$. The term $6xy$ divided by $2x$ gives $3y$. Therefore,

$(3y)^2$ or $9y^2$ must be the third term, and $x^2 + 6xy$ completed gives $x^2 + 6xy + 9y^2$.

6. Complete the square in the following expressions:

(a) $x^2 + 2x$
(b) $a^2 + 6a$
(c) $b^2 - 4b$
(d) $c^2 - 12c$
(e) $x^2 - 4xy$
(f) $a^2 - 8ac$
(g) $m^2 + 2mn$
(h) $3^2 + 4 \cdot 3$
(i) $(\frac{1}{2})^2 + 6(\frac{1}{2})$
(j) $a^2 + 2a$
(k) $a^2 + a$
(l) $p^2 - q$
(m) $x^2 + xy$
(n) $r^2 - rs$
(o) $a^4 - 2a^2b^2$
(p) $x^4 - x^2y^2$

7. (a) Since $(x + a)^2 = x^2 + 2ax + a^2$,

$$\sqrt{x^2 + 2ax + a^2} = x + a$$

(b) Is there another square root of $x^2 + 2ax + a^2$? If so, what is it?

(c) Evidently the square root of a perfect square is one of its two equal factors.

8. After completing the squares of the expressions in Ex. 6, give the square root of each trinomial.

E. SOLUTION OF A QUADRATIC EQUATION BY COMPLETING THE SQUARE

I. First Example

1. If the length of a square is increased by 4 ft., the area of the rectangle formed is 32 sq. ft. Find the dimensions of the rectangle.

Solution:

Let n = no. of ft. in side of □.
Then $n + 4$ = " " " " length of ▭.

and $\quad n(n + 4) =$ no. of sq. ft. in $S_\square$.
But $\quad 32 =$ " " " " " $S_\square$.

$$\therefore n(n + 4) = 32 \tag{1}$$

Expanding (1), $n^2 + 4n = 32$ (2)

Completing the $\square$, by adding 4 to each side of (2)

$$n^2 + 4n + 4 = 36 \tag{3}$$

Extracting the square root of each side of (3)

$$n + 2 = \pm 6 \tag{4}$$

Using $+ 6$, $\quad n = 4$ and $n + 4 = 8$ (5)

Using $- 6$, $\quad n = - 8$ and $n + 4 = - 4$ (6)

$$\therefore \square = 4 \text{ ft. by } 8 \text{ ft.} \tag{7}$$

Proof, $\quad S_\square = 4 \cdot 8 = 32$ sq. ft.

2. As in all quadratics, this equation has two roots. But, since the negative root is meaningless in this problem except in graphing, it may be dropped.

In all abstract problems and in all others, except where the negative root has no meaning, both roots are retained. Notice that the standard form of this equation, $n^2 + 4n - 32 = 0$, is not used in this method. Only the two terms with n^2 and n are kept in the first member. The other term is transposed to the second member.

II. Second Example

1. Solve $x^2 + 3x - 10 = 0$ (1)

Solution:

Transposing in (1),

$$x^2 + 3x = 10 \tag{2}$$

Completing the $\square$ by adding $\left(\frac{3}{2}\right)^2$, or $\frac{9}{4}$, in (2)

$$x^2 + 3x + \frac{9}{4} = \frac{49}{4} \tag{3}$$

Extracting the square root of each side of (3),

$$x + \frac{3}{2} = \pm\frac{7}{2} \qquad (4)$$

Using $+\frac{7}{2}$, $\quad x = \frac{4}{2}$ or $2 \qquad (5)$

Using $-\frac{7}{2}$, $\quad x = \frac{-10}{2}$ or $-5 \qquad (6)$

Roots, $\quad \therefore\ x = 2$ or $-5 \qquad (7)$

Checking:

$$\begin{array}{rcl|rcl} x^2 + 3x - 10 & = & 0 & x^2 + 3x - 10 & = & 0 \\ x & = & 2 & x & = & -5 \\ 4 + 6 - 10 & = & 0 & 25 - 15 - 10 & = & 0 \\ 0 & = & 0 & 0 & = & 0 \end{array}$$

III. Third Example

1. If the second degree term of a quadratic has a coefficient, make its coefficient 1, by dividing both members of the equation by the given coefficient.

2. *Solve:* $2x^2 - 3x = 9 \qquad (1)$

Divide both sides of (1) by 2, the coefficient of x

$$x^2 - \frac{3x}{2} = \frac{9}{2} \qquad (2)$$

Completing the square by adding $\frac{9}{16}$ to both sides,

$$x - \frac{3x}{2} + \frac{9}{16} = \left[\frac{9}{2} + \frac{9}{16}\right] = \frac{81}{16} \qquad (3)$$

Extracting the square root of both sides,

$$x - \frac{3}{4} = \pm\frac{9}{4} \qquad (4)$$

Collecting terms, $x = \frac{12}{4}$ or $3 \qquad (5)$

and $x = \frac{-6}{4}$ or $\frac{-3}{2} \qquad (6)$

Check:

$2x^2 - 3x = 9$	$2x^2 - 3x = 9$
$x = 3$	$x = \frac{-3}{2}$
$18 - 9 = 9$	$\frac{9}{2} + \frac{9}{2} = 9$
$9 = 9$	$9 = 9$

3. (a) Solve the following quadratics by the method of completing the square.
(b) Check the roots.

(1) $x^2 - 10x = -24$	(11) $e^2 - e = 30$
(2) $n^2 - 4n = 21$	(12) $x^2 - 10xy = -16y^2$
(3) $x^2 - 3x = 10$	(13) $d^2 + 8d + 15 = 0$
(4) $2x^2 - 5x = 25$	(14) $3x^2 - 2xy = 8y^2$
(5) $b^2 - 13b = -30$	(15) $3a^2 + 7ab - 6b^2 = 0$
(6) $t^2 - 6t - 16 = 0$	(16) $6x^2 - 5x = -1$
(7) $s^2 - 16s + 48 = 0$	(17) $4x^2 - 5xy = 6y^2$
(8) $x^2 - 4xy = 45y^2$	(18) $y^2 - yz = 56z^2$
(9) $c^2 + 6cx - 7x^2 = 0$	(19) $6a^2 + 5a + 1 = 0$
(10) $n^2 + 12n = 28$	(20) $3a^2 + 17ab - 6b^2 = 0$

IV. Fourth Example of a Quadratic Equation

In some quadratic equations, the second member is such a number that its square root is a surd. The following example shows how to solve such an equation.

$$x^2 + 2x - 7 = 0 \qquad (1)$$

Transposing -7, $\quad x^2 + 2x = 7 \qquad (2)$

Completing the square,

$$x^2 + 2x + 1 = 8 \qquad (3)$$

Extracting $\sqrt{}$ of each side of (3),

$$x + 1 = \pm\sqrt{8} = \pm 2\sqrt{2} \qquad (4)$$

Using $+2\sqrt{2}$, $\quad x = 2\sqrt{2} - 1 \qquad (5)$

Using $-2\sqrt{2}$, $x = -2\sqrt{2} - 1$ (6)

Decimal equivalents $x = 2 \times 1.41421 - 1 = 1.82842$ (7)

$x = -2 \times 1.41421 - 1 = -3.82842$ (8)

2. (a) In order to check such roots, one must know how to square such quantities as $2\sqrt{2} - 1$ and how to multiply such surds by other numbers.

(b) Checking the first root.

$$\begin{aligned}
\text{If } x &= 2\sqrt{2} - 1 \\
x^2 &= (2\sqrt{2} - 1)^2 \\
&= 4\cdot 2 - 4\sqrt{2} + 1 \\
&= 9 - 4\sqrt{2} \\
2x &= 2(2\sqrt{2} - 1) \\
&= 4\sqrt{2} - 2
\end{aligned}$$

(c) Substituting these values in

$x^2 + 2x = 7$, we have

$9 - 4\sqrt{2} + 4\sqrt{2} - 2 = 7$

The $-4\sqrt{2}$ and the $+4\sqrt{2}$ offset each other.

$\therefore 7 = 7$

NOTE. — In squaring $2\sqrt{2}$, remember that the coefficient 2, when squared, gives 4 and that $\sqrt{2}$, when squared, gives $\sqrt{2}^2$ which is 2. Therefore $(2\sqrt{2})^2 = 2^2\cdot 2 = 8$.

Squaring any quantity, that is wholly under a square root sign, may be done by merely removing the radical sign. The coefficient, however, must be multiplied by itself; as $(\sqrt{5})^2 = 5$; $(5\sqrt{5})^2 = 25\cdot 5 = 125$; $(\sqrt{x})^2 = x$; $(a\sqrt{x})^2 = a^2x$.

(d) Checking the second root,

$$\begin{aligned}
\text{If } x &= -2\sqrt{2} - 1 \\
x^2 &= (-2\sqrt{2} - 1)^2 \\
&= +4\cdot 2 + 4\sqrt{2} + 1 \\
&= 9 + 4\sqrt{2} \\
2x &= 2(-2\sqrt{2} - 1) \\
&= -4\sqrt{2} - 2
\end{aligned}$$

(e) Substituting these values for x^2 and $2x$ in the equation, we have $x^2 + 2x = 7$

$$9 + 4\sqrt{2} - 4\sqrt{2} - 2 = 7$$
$$7 = 7$$

(f) Is it possible to check the equation by using the decimal values for x? Why?

NOTE. — The pupil should know the decimal equivalents for $\sqrt{2}$, $\sqrt{3}$ and $\sqrt{5}$. For the values of other surds, use the table of square roots given in the Appendix.

3. (a) Solve the following equations.

(b) Find the decimal values of the roots, correct to three decimal places.

(1) $x^2 + 2x - 17 = 0$

(2) $x^2 + 4x + 1 = 0$

(3) $x^2 + 4x - 23 = 0$

(4) $x^2 + 2x - 4 = 0$

(5) $x^2 - 6x + 2 = 0$

(6) $x^2 - 4x = 2$

(7) $x^2 - 6x = 11$

(8) $b^2 - 2b = 5$

(9) $x^2 - 10x = -22$

(10) $a^2 - 6a = 1$

(11) $x^2 - 12x = 9$

(12) $x^2 - 14x - 7 = 0$

F. FOURTH METHOD OF SOLVING QUADRATIC EQUATIONS — BY FORMULA

I. Advantages of Formulas

1. You have learned how convenient a formula is. What are some of its advantages?

2. Its greatest advantage is that it states the results, not for one particular problem, but for all problems of its kind. In other words, it gives a general solution from which results for special problems may be found by substituting particular values.

For example, $2(lw + lh + wh)$ stands for the area of the surface of any rectangular solid. The numerical value of a special solid may be found by substituting its particular dimensions for l, w, and h, in the formula.

3. The equation, $ax^2 + bx + c = 0$, in the *general form* of a quadratic, for it represents all possible quadratic equations. If we solve this general form, its results will be general; that is, they will represent the values of x in all quadratics. These general results will be *formulas* for the *roots* of a *quadratic*, from which the roots of a particular equation may be found by substituting its own values of the coefficients a and b and the known term c.

II. Solution of the General Quadratic

1. The general quadratic must be solved as any other by completing the square. By solving this one, we are completing the square at one time for all quadratics.

(a) Solve $ax^2 + bx + c = 0$ (1)

Transposing c gives

$$ax^2 + bx = -c \tag{2}$$

Dividing both members by a, the coefficient of x^2, gives

$$x^2 + \frac{b}{a}x = -\frac{c}{a} \tag{3}$$

Adding $\left(\frac{b}{2a}\right)^2$, or $\frac{b^2}{4a^2}$, to complete the square, gives

$$x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} = \frac{b^2}{4a^2} - \frac{c}{a} \tag{4}$$

Adding the fractions in the second member of (4)

$$x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} = \frac{b^2 - 4ac}{4a^2} \tag{5}$$

Extracting the square root of both members of (5),

$$x + \frac{b}{2a} = \frac{\pm\sqrt{b^2 - 4ac}}{2a} \tag{6}$$

Transposing $\frac{b}{2a}$ gives

$$x = \frac{-b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \qquad (7)$$

One root is $\frac{-b + \sqrt{b^2 - 4ac}}{2a}$ and the other is

$$\frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

(b) In these formulas, a stands for the coefficient of x^2; b stands for the coefficient of x; and c is for the known term.

III. The Formula Used to Solve a Quadratic

1. (a) Solve $2x^2 - 13x = 45$

Solution:

$$2x^2 - 13x = 45 \qquad (1)$$

General form, $2x - 13x - 45 = 0$ (2)

$a = 2, b = -13, c = -45$ (3)

From formula, $x = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$ (4)

Substituting values,

$$x = \frac{-(-13) + \sqrt{(-13)^2 - 4 \cdot 2(-45)}}{2 \cdot 2} \qquad (5)$$

Expanding, $x = \frac{13 + \sqrt{169 + 360}}{4}$ (6)

$$= \frac{13 + \sqrt{529}}{4}$$

Collecting, $x = \frac{13 + 23}{4} = \frac{36}{4} = 9$ (7)

Using second formula,

$$x = \frac{-b - \sqrt{b^2 - 4ac}}{2a} \tag{8}$$

$$= \frac{13 - \sqrt{529}}{4} \tag{9}$$

$$= \frac{13 - 23}{4} = \frac{-10}{4} = -\frac{5}{2} \tag{10}$$

$$x = 9 \text{ or } -\frac{5}{2} \tag{11}$$

(b) After a little practice the double sign, $\pm$, may be used and both roots found at once.

(c) This method may be shortened by making several of these steps mentally.

(d) Check these roots in the equation.

2. (a) Solve the following quadratic equations by formula.

(b) Check results.

(1) $x^2 - 9x + 20 = 0$
(2) $x^2 - 8x + 12 = 0$
(3) $x^2 - x = 30$
(4) $x^2 + 5x - 24 = 0$
(5) $x^2 + 7x - 30 = 0$
(6) $x^2 - 14x + 45 = 0$
(7) $3x^2 + x - 2 = 0$
(8) $2x^2 + 11x - 6 = 0$
(9) $x^2 - 5x = 50$
(10) $x^2 - 7x + 12 = 0$
(11) $x^2 - 10x = -21$
(12) $3x^2 + 2x = 8$
(13) $4x^2 - 11x + 6 = 0$
(14) $2x^2 - 9x + 4 = 0$

G. PROBLEMS IN QUADRATIC EQUATIONS

The following problems give quadratic equations. Change each to standard form and solve. Vary the methods. Check roots.

1. A group of boys and girls decided to make a tennis court. How much wire was needed to put across the ends of the court if the length was 6 ft. more than twice the width and the area, 2808 sq. ft.?

2. A man has a farm facing a certain road. He wishes

to give to each of his three sons a field facing this road, containing 468 sq. rd. Each field has a length 8 rods greater than the width. What is the length and width of each?

3. The denominator of a fraction is 3 more than the numerator. The fraction added to its reciprocal equals $\frac{29}{10}$. Find the fraction.

4. (a) Draw a line of any length. Call it x inches. Draw a rectangle with a length of $x + 3$ inches and a width of $x + 2$ inches. How long must x be if the area of the rectangle is to be 56 sq. in.?

(b) How long must x be if the area of the rectangle is to be 90 sq. in.? 30 sq. in.? 156 sq. in.? 210 sq. in.?

5. The sum of two numbers is 19 and their product is 70. What are the numbers?

6. The difference between two numbers is 5 and their product is 84. Find the numbers.

7. A farmer wishes to give his son a field containing 40 acres. Find the width of the field if it is to be 36 rds. longer than wide.

8. The product of two consecutive whole numbers is 342. Find the numbers.

9. One leg of a right triangle is 2 ft. more than the other and the hypotenuse is 10 ft. long. Find the length of each leg.

10. One side of a rectangle is 2 in. less than the diagonal and the other is 9 in. less. How long is the diagonal?

11. One side of a rectangle is 4.5 in. less than the diagonal and the other is 1 in. less. Find the length of the diagonal.

12. One side of a rectangle is 2 ft. less than the diagonal and the other is 25 ft. less. Find the length of the diagonal.

13. One leg of a right triangle is 17 ft. longer than the other. The hypotenuse is 25 ft. How long is each leg?

14. A field is in the shape of a right triangle, whose hypotenuse is 74 ft. long. One leg is 46 ft. longer than the other. What is the perimeter of the field? the area?

15. The hypotenuse of a right triangle is 4.25 in. One leg is 1.75 in. shorter than the other. Find the length of the legs.

16. The sum of the squares of two consecutive numbers is 145. Find the numbers.

17. The sum of the squares of two consecutive odd numbers is 74. Find the numbers.

18. The sum of the squares of two consecutive even numbers is 100. Find the numbers.

19. The area of a triangle is 54 sq. ft. The altitude is 3 ft. less than the base. Find the altitude and the base.

20. When a body is thrown vertically upward, its height at the end of a given number of seconds may be found approximately from the formula, $h = a + vt - 16t^2$, in which h is the height of the body at the end of t seconds, v is the velocity in feet a second with which it starts, and a is the height in feet above the earth of the position from which it starts.

(a) How long will it take a sky rocket to reach a height of 692 ft., if it starts vertically upward with a velocity of 208 ft. a second, from a platform 16 ft. high?

(b) A bullet is fired vertically from the level of the ground with a velocity of 1040 ft. a second. In how many seconds will it reach a height of 16,900 ft.?

(c) A bullet is fired vertically upward from a cliff 50 ft. high. If it has an initial velocity of 600 ft. a second, in how many seconds will it reach a height of 5675 ft.?

CHAPTER TEN

QUADRATIC EQUATIONS WITH TWO UNKNOWNS

A. SYSTEMS OF QUADRATIC EQUATIONS

1. What are simultaneous linear equations?

2. In a system of equations, upon what does the necessary number of equations depend?

3. Some problems with two unknowns may give equations of the second degree. To solve such a problem, it must give as many equations as it has unknown quantities.

4. A system of equations, one or more of which is of the second degree, is a system of *simultaneous quadratic equations.*

I. Solution by Graphs

1. *Solve the following problem:*

The sum of two numbers is 20. The square of the first number equals the second. Find the numbers.

(a) *Solution:*

Let x = one number
and y = the other number.

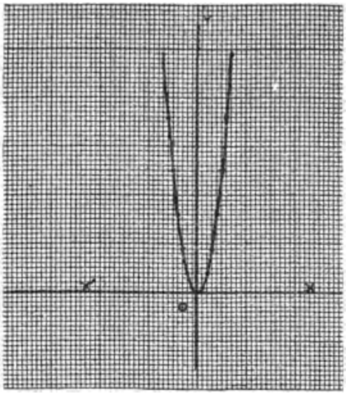
The statements in the problem give the equations

$$x + y = 20 \qquad (1)$$

$$x^2 = y \qquad (2)$$

(b) Which of these equations is quadratic?

(c) Make a table of possible values for x and y in $x^2 = y$, plot enough points to determine its curve, and draw its graph. What is the name of this curve?



GRAPH OF $x^2 = y$

$x^2 = y$

If x is	then y is
0	0
1	1
2	4
3	9
4	16
5	25
7	49
− 1	1
− 2	4
− 3	9
− 4	16
− 5	25
− 7	49

(d) Of what degree is the equation $x + y = 20$? What is the shape of its graph? How many of its points must be located to determine its position?

$x + y = 20$

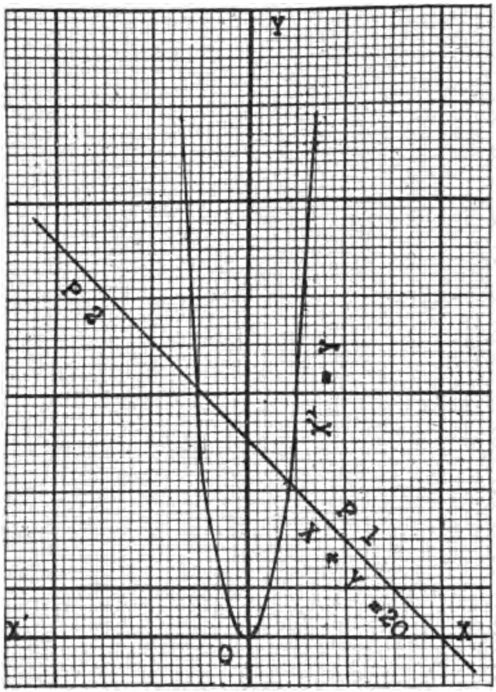
If x is	then y is
0	20
20	0

(e) On the same pair of axes, draw the graphs of both equations.

(f) The straight line cuts the curve in two points P_1 and P_2.

(g) The location of P_1 is (4, 16) and of P_2 is (− 5, 25). From the coördination of these points, we get two sets of values for x and y. When x is 4, y is 16, and when x is − 5, y is 25.

(h) Check each set of roots in both equations.



2. (a) Every quadratic does not have a graph in the shape of a parabola. The next solution shows a second kind of curve.

(b) Graph the following system:

$$x^2 + y^2 = 25 \qquad (1)$$

$$x + y = 7 \qquad (2)$$

From (1), $y = \pm\sqrt{25 - x^2}$

Compute the value of y, when x changes from $+6$ to -6.

$x^2 + y^2 = 25$

When x is	then y is
0	± 5
1	$\pm 2\sqrt{6} = \pm 4.9$
2	$\pm \sqrt{21} = \pm 4.6$
3	± 4
4	± 3
5	0
6	$\pm \sqrt{-11}$ *
-1	± 4.9
-2	± 4.6
-3	± 4
-4	± 3
-5	0
-6	$\pm \sqrt{-11}$ *

$x + y = 7$

If x is	then y is
0	7
7	0

* It is impossible to take the square root of a negative number. Such an indicated root is called an *imaginary number*.

Why is it unnecessary to assign values to x higher than $+6$ or lower than -6?

(c) What is the shape of the curve of $x^2 + y^2 = 25$?
(d) What is the radius of the curve?
(e) How does the radius compare with the second member of the equation?
(f) At what points does the line graph cut the circle?
(g) From the coördinates of these two points we get

$$x = 4 \text{ and } y = 3, \text{ or } x = 3 \text{ and } y = 4.$$

(h) Check in both equations.

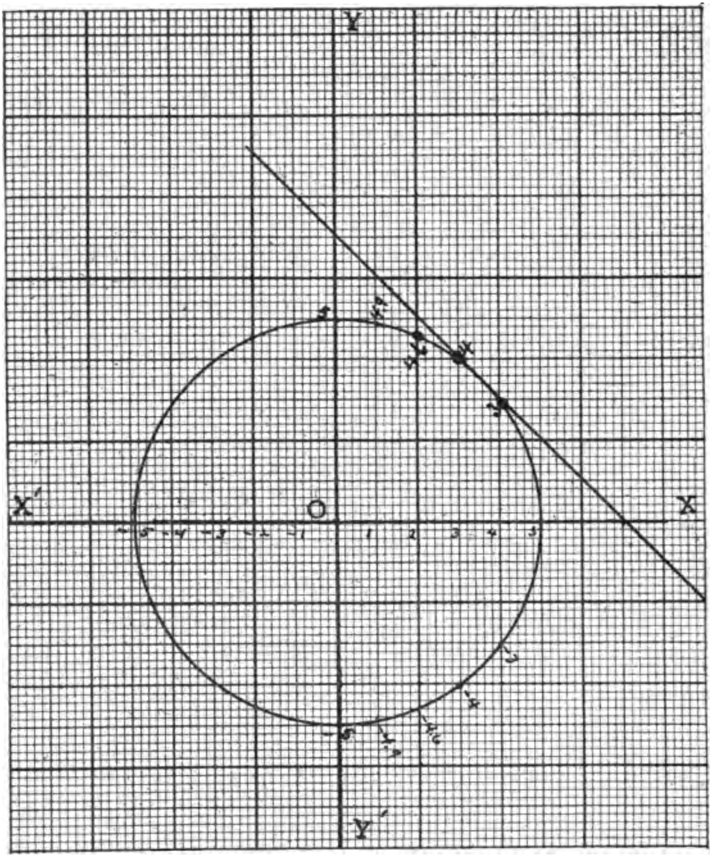
3. (a) Solve the following systems by graphs.
(b) Check the roots.

(1) $\begin{aligned} x^2 + y^2 &= 100 \\ x - y &= 2 \end{aligned}$ (2) $\begin{aligned} x^2 + y^2 &= 169 \\ x + y &= 17 \end{aligned}$

4. Other quadratic equations give curves that are different from the parabola and circle. For illustrations, see the Appendix.

II. Solution by Substitution

1. A system of equations, one of which is quadratic, may be solved easily by substitution.



GRAPH OF $x^2 + y^2 = 25$ AND $x + y = 7$

2. (a) Solve the system

$$3x^2 - 4y^2 = 8 \qquad (1)$$

$$5x - 4y = 10 \qquad (2)$$

From (2), $$x = \frac{10 + 4y}{5} \qquad (3)$$

Substituting (3) in (1),

$$\frac{3(100 + 80y + 16y^2)}{25} - 4y^2 = 8 \qquad (4)$$

Multiplying (4) by 25, and expanding,

$$300 + 240y + 48y^2 - 100y^2 = 200 \qquad (5)$$

Collecting terms,

$$-52y^2 + 240y + 100 = 0 \qquad (6)$$

Dividing (6) by -4,

$$13y^2 - 60y - 25 = 0 \qquad (7)$$

Factoring (7),

$$(13y + 5)(y - 5) = 0 \qquad (8)$$

Solving (8),

$$y = -\frac{5}{13} \text{ or } 5 \qquad (9)$$

Substituting (9) in (3)

$$x = \frac{10 + 4y}{5} \qquad (3)$$

When $y = -\dfrac{5}{13}$ (9)

$$\text{then, } x = \frac{10 - \frac{20}{13}}{5} \qquad (10)$$

$$= 2 - \frac{4}{13}$$

$$= \frac{22}{13}$$

$$x = \frac{10 + 4y}{5} \qquad (3)$$

When $y = 5$ (9)

$$\text{then, } x = \frac{10 + 20}{5} \qquad (11)$$

$$= 6$$

Therefore $\left.\begin{array}{l} x = 6 \\ \text{and } y = 5 \end{array}\right\}$ or $\left.\begin{array}{l} x = \dfrac{22}{13} \\ \text{and } y = \dfrac{5}{13} \end{array}\right\}$

(b) Check each set of roots in equations (1) and (2).

3. (a) Solve the following systems of simultaneous quadratic equations.

(b) Check results.

(1) $x + y = 17$, $x = y^2 - 13$

(2) $x^2 + y^2 = 41$, $x - y = 1$

(3) $x^2 + y^2 = 53$
$x + y = 9$

(4) $4x^2 - 3y^2 = 24$
$7x - 3y = 15$

(5) $x^2 - y^2 = 21$
$x + y = 3$

(6) $3a^2 - b^2 = 32$
$a + 2b = 12$

(7) The sum of two numbers is 16. The sum of their squares is 136. Find the numbers.

(8) The area of a schoolroom floor is 825 sq. ft. The perimeter is 116 ft. Find the dimensions.

(9) The sum of the areas of two square fields is 100 sq. rd. The difference between the lengths of their sides is 2 rd. Find the side of each.

(10) The area of a square on the hypotenuse of a right triangle is 225 sq. ft. The base of the right triangle is 3 ft. longer than the altitude. Find the two legs of the right triangle.

CHAPTER ELEVEN

RATIOS IN RIGHT TRIANGLES

A. PROPERTIES OF SIMILAR FIGURES

1. What are similar figures?

2. Compare corresponding or homologous angles in similar figures.

3. What is the relation between pairs of homologous sides in similar figures?

4. Under what conditions are two triangles similar?

5. Under what conditions are two right triangles similar?

6. How did Thales find the height of an Egyptian pyramid from his staff? (See Book Two.)

7. How can you find the height of a tree from a stick and the shadows?

B. RATIO OF SIDE OPPOSITE TO SIDE ADJACENT TO AN ACUTE ANGLE

I. Constant Value of the Ratio for Each Angle

1. (a) DEF is a right triangle made by a tree, its shadow, and the rays of the sun. ABC is a right triangle similarly made by a stick and its shadow.

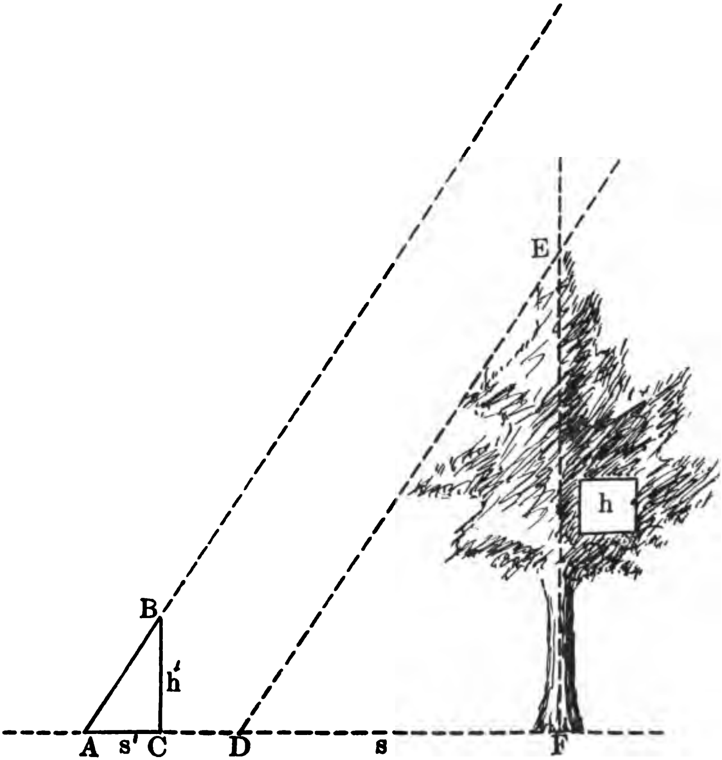
(b) How can you prove that $\triangle ABC$ is similar to $\triangle DEF$?

(c) Write a proportion that holds between the heights and the shadows.

(d) Find the height of the tree, if its shadow is 21 ft., the stick is 5 ft., and its shadow is $3\frac{1}{2}$ ft.

(e) Measure the angle A with your protractor. Test angle D to make sure that $\angle D = \angle A$.

(f) In $\triangle ABC$ find the ratio of h' to s', carrying the result to three decimal places.

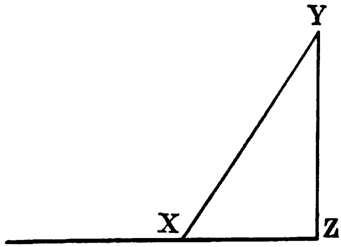


(g) Find the ratio of the height of the tree to its shadow, correct to three decimal places.

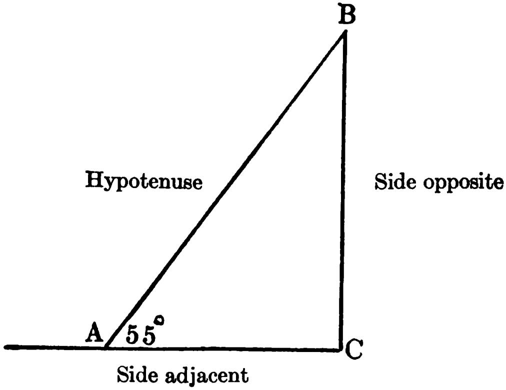
2. (a) On squared paper draw a large right triangle XYZ with $\angle X$ equal to $\angle A$ or $\angle D$. Letter the right angle Z.

(b) Carefully measure YZ and XZ, and find their ratio correct to three decimal places.

3. (a) How do the decimal values of these three ratios compare?
   (b) They should be approximately equal to 1.4. If it were not for inaccuracies in drawing and measurement, they would be 1.428.

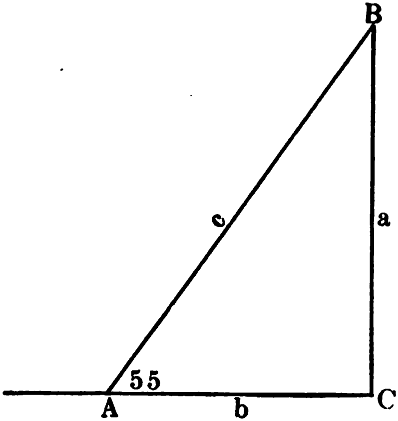


4. Each of these ratios expresses the relation between the side *opposite* the angle of 55° to the side *adjacent* to the angle.



5. Mathematicians find it convenient to letter the two acute angles of a right triangle A and B, and the right angle C. The side opposite each angle has the same letter as the angle, but a small or algebraic letter instead of a capital or geometric letter.

6. Since all right triangles having one acute angle equal to 55° are similar triangles, and in similar triangles, hom-



ologous sides have the same ratio, we know that in any right triangle, $\frac{\text{the side opposite } 55°}{\text{the side adjacent to } 55°}$ i.e. $\frac{a}{b} = 1.428 +$.

7. (a) Draw two right triangles of different sizes, each of which has an angle of 30°.
   (b) Carefully measure the *side opposite* $\angle 30°$ and the *side* adjacent to $\angle 30°$ in each, and find the value of each ratio correct to two decimal places.
   (c) How do the two ratios compare?

8. Do the same with two right triangles having an acute angle of 45°.

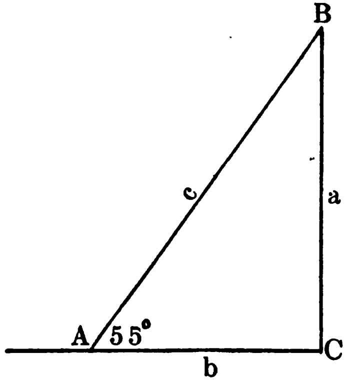
9. We see, therefore, for each acute angle of a right triangle the ratio of the *side opposite* the acute angle to the *side adjacent* to the angle has a fixed and definite value, regardless of the size of the triangle.

II. Name of the Ratio

1. This ratio of the *side opposite* an *acute angle* to the *side adjacent to the same angle* is such an important one that we

give it a special name. It is important because with it we can find the height of a tree or other unknown distance, from one side of one triangle, instead of needing the measures of three lines in two triangles, as is necessary in computing from similar triangles.

2. (a) The name of this ratio is the *tangent* of the angle. It is abbreviated *tan.*



(b) Tan A = tan 55° $= \frac{a}{b} = 1.428 +$.

III. Tangent Represented Geometrically

1. (a) On squared paper draw a circle one inch in radius.

(b) At one end of a diameter, draw a line perpendicular to it. This line just touches the circle and is called a *tangent* to the circle. The word *tangent* comes from a Latin word which means *touching.*

(c) At the center of the circle make an angle of 55°, prolonging the radius until it meets the tangent.

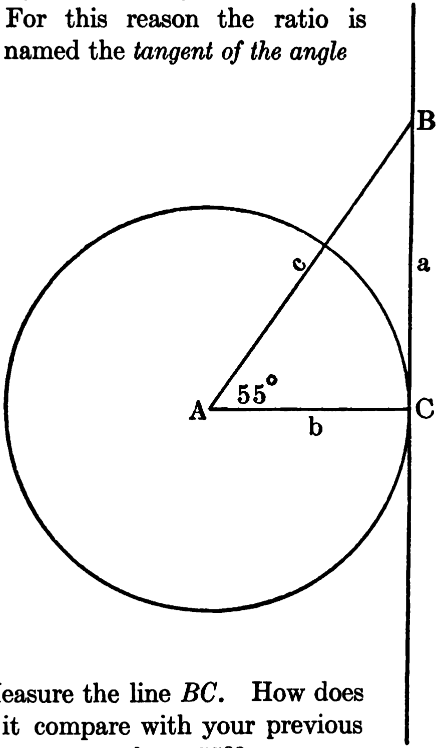
In the right $\triangle ABC$,

$$\tan A = \frac{a}{b}.$$

Since the radius, or b, is 1,

$$\tan A = \frac{a}{1} = a.$$

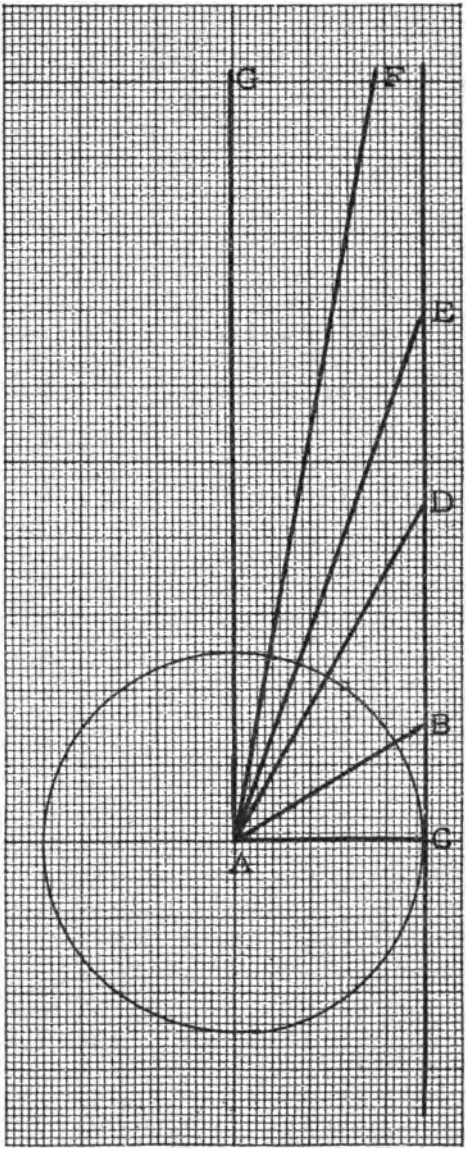
(d) We see, therefore, that in a circle whose radius is one unit, the ratio of *side opposite* to the *side adjacent* to the angle at the center is represented by the line *tangent to the circle.* For this reason the ratio is named the *tangent of the angle*



(e) Measure the line *BC*. How does it compare with your previous measures of tan 55°?

2. (a) On squared paper, draw a circle, one inch in diameter. Draw a line tangent to the circle at the end of the diameter. (This line may be drawn perpendicular to the diameter.)

(b) At the center draw angles of 30°, 60°, 70°, 80°. Prolong the sides of these angles to meet the tangent if possible.


G
F
E
D
B
A
C

(c) Why does the measure of BC give the tangent of 30°?

(d) Which line represents tan 60°?

(e) Is tan 60° twice as long as tan 30°?

(f) Why is it impossible to draw a tangent of 90°? When the angle at the center increases to 90°, the side AG becomes parallel to the tangent line; that is, the tangent becomes infinitely long. Any number that is infinitely large is called *infinity* and is represented by the sign ∞. Since the sign has no end, it is an appropriate symbol for infinity.

(g) Carefully measure the lengths of the tangents of angles 30°, 60°, and 70°. These measures should approximate the tangents given in the following table.

TABLE OF TANGENTS

Angle	Tan	Angle	Tan	Angle	Tan
0°	0	. . .	. . . . . .	. . .	. . . . . .
10°	0.1763	40°	0.8391	70°	2.7475
20°	0.3640	50°	1.1918	80°	5.6713
30°	0.5774	60°	1.7321	90°	∞

IV. Finding Heights or Distances by Tangents

1. (a) A tree makes a shadow 15 ft. long. Sighting from the end of the shadow to the top of the tree, a boy finds the angle of elevation to be 70°. How tall is the tree?

(b) *Solution:*

The formula for the tangent is,

$$\frac{a}{b} = \tan A. \qquad (1)$$

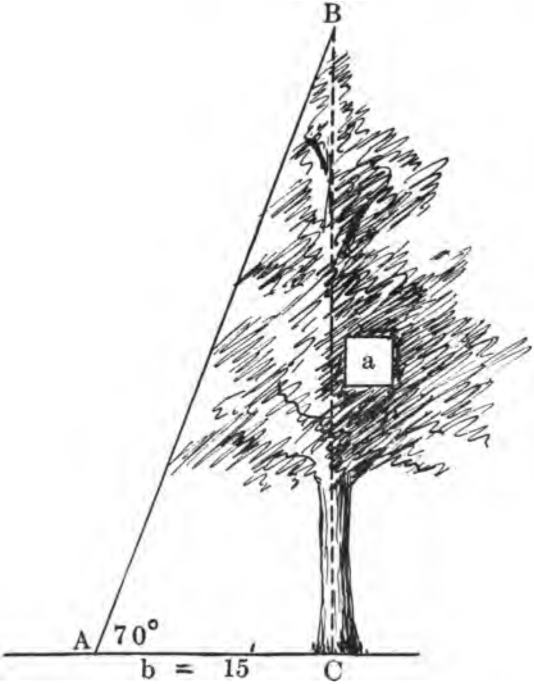
Multiplying both sides by b,

$$a = b \tan A. \qquad (2)$$

We have given that $b = 15$ ft. and $\angle A$ is 70°. From the table, we find $\tan 70° = 2.7475$. Substituting these known values in (2), we have

$$a = 15 \cdot 2.7475 \qquad (3)$$
$$= 41.2125$$

The height of the tree is approximately $41\frac{1}{4}$ ft.



(c) Since we do not have tape lines that divide a foot into such small decimals, it is absurd to keep such a result as 41.2125 ft. The decimal

part of a foot, .2125 ft., reduced to inches gives between $2\frac{1}{2}$ and 3 inches or nearly $\frac{1}{4}$ of a foot.

2. A tree makes a shadow 20 ft. long. Sighting from the end of the shadow to the top of the tree, a boy finds the angle of elevation to be 60°. How tall is the tree?

3. A flagpole makes a shadow 42 ft. long. The angle of elevation from the end of the shadow to the top of the pole is 20°. How tall is the pole?

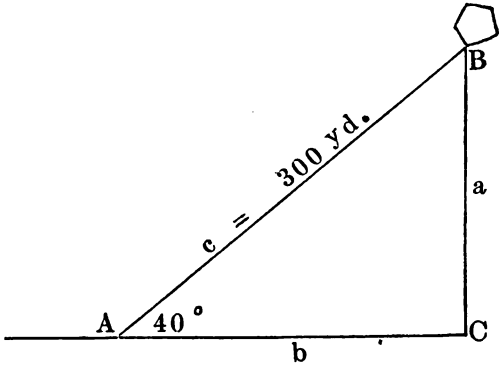
4. A pole casts a shadow 3 ft. long. The angle of elevation from the end of the shadow to the top of the pole is 80°. Find the height of the pole.

5. A boy stands 120 ft. from a school building. He finds the angle of elevation from his eye to the top of the building to be 30°. If his eye is 5 ft. from the ground, how high is the building?

C. RATIO OF THE SIDE OPPOSITE AN ACUTE ANGLE TO THE HYPOTENUSE

I. Finding the Ratio for an Angle of 40°

1. In a right triangle, there are other ratios besides that of the *side opposite* to the side *adjacent*, or the tangent.



2. (a) The length of a kite string is 300 yd. and the angle of elevation of the kite is 40°. Find the height

of the kite, supposing the line of the kite string to be straight.

(b) In this problem we wish to know the length of a, the *side opposite* the known angle. But we do not know the length of the *side adjacent*. Therefore, the ratio of the tangent will not help us to solve the problem.

(c) Since the *hypotenuse* is known, if we knew the value of the ratio of the *side opposite* an angle of 40° to the *hypotenuse*, we could find the height of the kite.

3. (a) This ratio of $\dfrac{\text{the side opposite an angle of } 40°}{\text{hypotenuse}}$ may be found approximately by drawing several right triangles with an angle of 40° and by measuring in each the *side opposite* and the *hypotenuse* and finding the value of the ratio.

(b) Why must these ratios be approximately the same, regardless of the size of the triangles? The ratios should be nearly .65.

II. Name of the Ratio

1. (a) The name given to this second ratio, that is, the ratio between the *side opposite an acute angle* and the *hypotenuse* of a *right triangle*, is the *sine* of the angle.

(b) The *sine* of an *angle A* is abbreviated *sin A*.

(c) The sine of any acute angle is always the ratio of the *side opposite* the angle to the *hypotenuse*.

TABLE OF SINES

Angle	Sin	Angle	Sin	Angle	Sin
0°	0	...	......	...	......
10°	0.1736	40°	0.6428	70°	0.9397
20°	0.3420	50°	0.7660	80°	0.9848
30°	0.5000	60°	0.8660	90°	1.0000

2. (a) In the problem about the kite, we may find its height by using the value of the sin 40° given in the table.

(b) The formula for the sin is

$$\frac{a}{c} = \sin A. \qquad (1)$$

Multiplying both sides by c, we have

$$a = c \sin A. \qquad (2)$$

We have given that $c = 300$ and $\angle A = 40°$.

$$\begin{aligned} a &= 300 \times .6428. \qquad (3)\\ &= 192.84 \text{ yd.}\\ &= 192 \text{ yd. } 2\tfrac{1}{2} \text{ ft. approximately}\\ &= 577\tfrac{1}{2} \text{ ft.} \end{aligned}$$

III. Sine Represented Geometrically

1. The sine of an angle, as well as its tangent, may be represented by the length of a line, if the angle is drawn at the center of a circle whose radius is one unit.

2. (a) On squared paper draw a circle 1 inch in diameter.

(b) At the center make an angle of 30°, as $\angle OAB$.

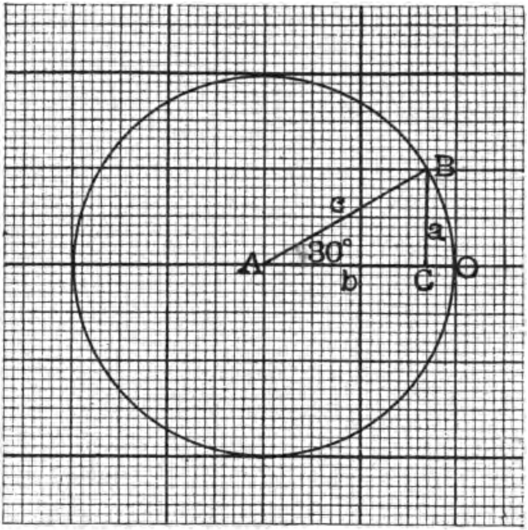
(c) From B, draw $BC \perp AO$.

(d) Evidently $\dfrac{BC}{AB} = \sin A = \dfrac{a}{c}$.

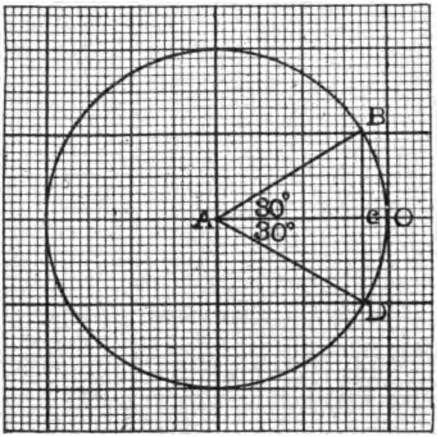
(e) But AB, or c, is the radius of the circle and equals 1.

$$\therefore \sin A = \frac{a}{c} = \frac{a}{1} = a.$$

(f) In a circle, whose radius is a unit, the length of the side opposite the acute angle is the measure of the sine.



3. (a) In the same circle, draw an angle of 30° below the radius AO as $\angle DAO$.
   (b) What kind of a triangle is $\triangle DAB$?



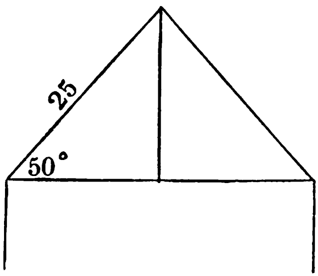
(c) How does BD compare with AB?
(d) How does BC compare with BD?
(e) What part of AB is BC?
(f) How long is BC?
(g) How does this value of the sin of $\angle 30°$ compare with that given in the table?

4. In a circle with an inch radius, draw angles of 20°, 45°, 60°, 80°, 90°.

5. How does the sine of an angle change as the angle increases from 0° to 90°?

IV. Finding Heights or Distances by Sines

1. A boy scout puts up a circular tent. He fastens the top by ropes 30 ft. long to stakes set in the ground. If the ropes are placed at an angle of 40° with the ground, how high should his vertical central pole be?



2. If the rafters of a gable roof are 25 ft. long and the pitch is 50°, what is the height of the ridge pole?

3. A ladder 16 ft. long has one end resting against a second story window. The other end makes an angle of 60° with the ground. How high is the window above the ground?

4. The length of a kite string is 350 yd. and the angle of elevation of the kite is 30°. Find the height of the kite.

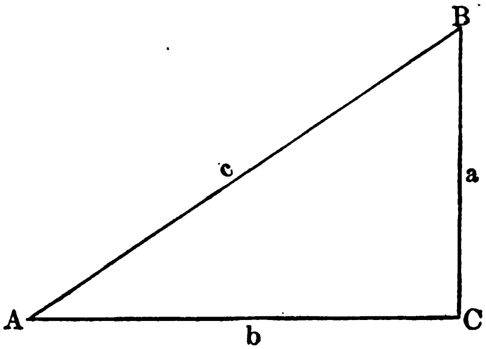
D. RATIO OF THE SIDE ADJACENT TO AN ACUTE ANGLE TO THE HYPOTENUSE

I. The Name of the Sine of the Complement of an Angle

1. Since a right triangle has three sides, there is a third possible ratio between them.

2. (a) We have used the ratio of $\frac{a}{b}$, or tan A, and the ratio of $\frac{a}{c}$, or sin A.

(b) Another possible ratio is $\frac{b}{c}$, that is, the ratio of the *side adjacent* to $\angle A$ to the *hypotenuse*.



(c) Since b is the *side opposite* $\angle B$, $\frac{b}{c}$ is sin B. The $\angle B$ is the *complement of* $\angle A$. Therefore, the ratio $\frac{b}{c}$ is called the *cosine* of $\angle A$. It is abbreviated *cos* A.

(d) The *sine* of one *acute angle* is the *cosine* of the *complement* of the *angle*.

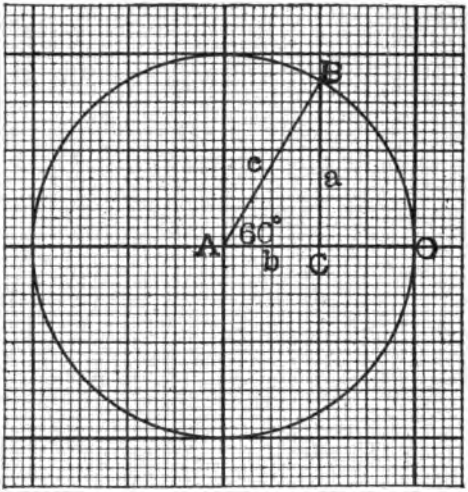
3. (a) From the table of sines find the cosines of each angle given, by reading the sine of its *complement*.

(b) Make a table of cosines.

II. Cosine Represented Geometrically

1. The cosine of an angle may be represented by the length of a line, if the angle is drawn at the center of a circle whose radius is one unit.

2. (a) On squared paper, draw a circle whose radius is 1 inch.
   (b) At the center make an angle of 60°, as $\angle OAB$.
   (c) From B, draw $BC \perp AO$.



   (d) Evidently $\frac{AC}{AB} = \cos A = \frac{b}{c}$.
   (e) But AB, or c, is the radius of the circle and equals 1.

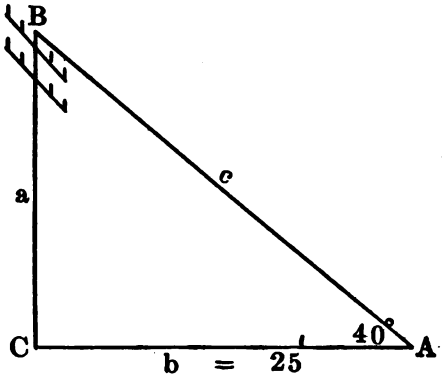
$$\therefore \cos A = \frac{b}{c} = \frac{b}{1} = b.$$

   (f) In a circle, whose radius is a unit, the length of the side adjacent to the angle measures the cosine.

3. (a) How does the length of AC compare with AB?
   (b) How long is AC?
   (c) Draw BO.
   (d) What kind of a triangle is OAB?
   (e) How does BC cut AO?
   (f) By the Pythagorean Theorem, find the length of BC.
   (g) What ratio does BC represent for angle 60°?
   (h) For what angle is BC the sine?

III. Finding Lengths and Distances by Cosines

1. (a) A wire, which steadies a telegraph pole, reaches from the top of the pole to a stake in the ground 25 ft. from the foot of the pole. It makes an angle of 40° with the ground. How long is the wire?



(b) *Solution:*

The formula for the cosine is

$$\frac{b}{c} = \cos A. \tag{1}$$

Multiplying both sides by c, we have

$$b = c \cos A. \tag{2}$$

Dividing both sides by $\cos A$, we have

$$\frac{b}{\cos A} = c. \tag{3}$$

Substituting known values for b and $\cos A$.

$$\frac{25}{.766} = c. \tag{4}$$

$$c = 32.7 + \text{ft.}$$

2. By using the formula for the tangent, find the height of the pole.

3. By using the Pythagorean Theorem, prove that your results are approximately correct.

4. A wire, which steadies a pole, is fastened to its top and reaches to a stake in the ground 18 ft. from the foot of the pole. It makes an angle of 60° with the ground. How long is the wire?

5. The streamers of a May pole made an angle of 50° with the floor when stretched to a point 12 ft. from the foot of the pole.

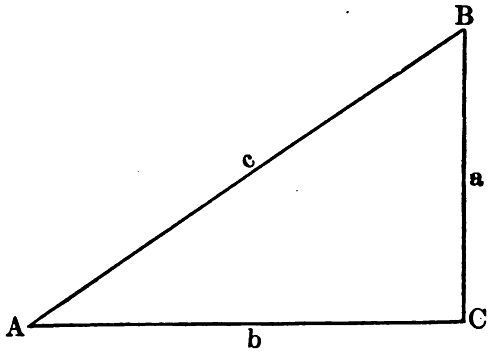
(a) How long were the streamers?

(b) How high was the pole?

6. A boy, wishing to find the height of his school building, found with his protractor that, at a distance of 30 ft., the angle of elevation from his eye to the top of the building was 50°. If the boy's eye was 5 ft. above the ground, how high was the building? Find the line of sight.

E. TANGENT OF THE COMPLEMENT OF AN ANGLE

I. The Name of the Reciprocal of the Tangent



1. In the triangle ABC, what ratio is the tangent of angle A? What is the tangent of angle B?

2. What is the relation of angle A to angle B?

3. (a) We see that the tangent of $\angle B$, $\frac{b}{a}$, is the inverted form of tangent of $\angle A$, $\frac{a}{b}$. Since the *reciprocal* of a number is *one divided* by the number, the *inverted* form of a *fraction* is its *reciprocal.*
   (b) The tangent of the *complement* of an angle is called the *cotangent* of the angle.
   (c) Cotangent of angle A is abbreviated cot A.
   (d) The cotangent of an angle is the reciprocal of its tangent.
4. (a) The tangent of 30° is .5774. What is the cotangent of 60°?
   (b) Tan 60° must equal the reciprocal of tan 30° or $\frac{1}{.5774}$. Divide 1 by .5774 and compare your result with the tangent of 60° as given in your table of tangents.
5. What is the cotangent of 20°; of 50°; of 70°; of 80°?

F. TRIGONOMETRIC FUNCTIONS

1. Each of these ratios, $\frac{a}{b}$, $\frac{b}{a}$, $\frac{a}{c}$, and $\frac{b}{c}$, changes in value as the size of the angle changes. Therefore, the *ratios* of *tan A*, *cot A*, *sin A*, and *cos A* are called *functions* of $\angle A$.
2. (a) There are two other ratios, or functions, of $\angle A$ possible. These are $\frac{c}{a}$ and $\frac{c}{b}$. These, too, have special names, which you will learn when you study *trigonometry.*
   (*b*) The word *trigonometry* comes from three Greek words and means *measure* of a *triangle.* The first part, *tri*, means *three;* the middle part comes from the word *gonia,* meaning angle; and the last you recognize as similar to the word *meter,*

I. Table of Trigonometric Functions

Angle	sin	cos	tan	cot	
0°	0.0000	1.0000	0.0000	∞	**90°**
1°	.0175	.9998	.0175	57.2900	89°
2°	.0349	.9994	.0349	28.6363	88°
3°	.0523	.9986	.0524	19.0811	87°
4°	.0698	.9976	.0699	14.3007	86°
5°	.0872	.9962	.0875	11.4301	**85°**
6°	.1045	.9945	.1051	9.5144	84°
7°	.1219	.9925	.1228	8.1443	83°
8°	.1392	.9903	.1405	7.1154	82°
9°	.1564	.9877	.1584	6.3138	81°
10°	.1736	.9848	.1763	5.6713	**80°**
11°	.1908	.9816	.1944	5.1446	79°
12°	.2079	.9781	.2126	4.7046	78°
13°	.2250	.9744	.2309	4.3315	77°
14°	.2419	.9703	.2493	4.0108	76°
15°	.2588	.9659	.2679	3.7321	**75°**
16°	.2756	.9613	.2867	3.4874	74°
17°	.2924	.9563	.3057	3.2709	73°
18°	.3090	.9511	.3249	3.0777	72°
19°	.3256	.9455	.3443	2.9042	71°
20°	.3420	.9397	.3640	2.7475	**70°**
21°	.3584	.9336	.3839	2.6051	69°
22°	.3746	.9272	.4040	2.4751	68°
23°	.3907	.9205	.4245	2.3559	67°
24°	.4067	.9135	.4452	2.2460	66°
25°	.4226	.9063	.4663	2.1445	**65°**
26°	.4384	.8988	.4877	2.0503	64°
27°	.4540	.8910	.5095	1.9626	63°
28°	.4695	.8829	.5317	1.8807	62°
29°	.4848	.8746	.5543	1.8040	61°
30°	.5000	.8660	.5774	1.7321	**60°**
31°	.5150	.8572	.6009	1.6643	59°
32°	.5299	.8480	.6249	1.6003	58°
33°	.5446	.8387	.6494	1.5399	57°
34°	.5592	.8290	.6745	1.4826	56°
35°	.5736	.8192	.7002	1.4281	**55°**
36°	.5878	.8090	.7265	1.3764	54°
37°	.6018	.7986	.7536	1.3270	53°
38°	.6157	.7880	.7813	1.2799	52°
39°	.6293	.7771	.8098	1.2349	51°
40°	.6428	.7660	.8391	1.1918	**50°**
41°	.6561	.7547	.8693	1.1504	49°
42°	.6691	.7431	.9004	1.1106	48°
43°	.6820	.7314	.9325	1.0724	47°
44°	.6947	.7193	.9657	1.0355	46°
45°	.7071	.7071	1.0000	1.0000	**45°**
	cos	sin	cot	tan	Angle

meaning *measure*. Trigonometry is a branch of mathematics, used largely in surveying and engineering, that is based chiefly on the measurement of triangles.

II. Explanation of the Table of Functions

The table of functions on the preceding page shows the sine, cosine, tangent, and cotangent of every degree of an angle between 0° and 90°. We have seen that the sine of an angle equals the cosine of the complement, and the tangent of an angle equals the cotangent of its complement. In general, the function of an angle equals the cofunction of its complement. Therefore, it is necessary to give the functions of angles only from 0° to 45°, for the functions of angles 45° to 90° may be found by reading the cofunction of angles from 0° to 45°.

III. Reading from the Table

1. To Find the Function of an Angle

(a) The left-hand column contains the number of degreesfrom 0° to 45°. To find the functions of any of these angles, look for the desired function at the *top* of the table and read the number in that column opposite the required angle. For example, to find tan 25°, read the number in the third column, *under* tan, opposite 25°, namely, .4663.

(b) The right-hand column begins at the *bottom* with 45° and numbers the angles *up* to 90°. This arrangement places each angle in the left-hand column opposite its complement in the right-hand column.

To find a function of any angle between 45° and 90°, look for the desired function at the *bottom* of the table and read the number in that column

opposite the required angle. For example, to find tan 55°, read the number in the fourth column, *above* tan, opposite 55°, namely 1.4281.

2. To Find the Angle from the Function

(a) Given cos A = .8192, to find $\angle A$.

Since the first two columns give the sines and cosines, the number .8192 must be found in one of these two columns. We find it in the second column, which has *cos* at the *top*. Therefore, the size of the angle must be read from the left-hand column. Since .8192 is opposite 35°, .8192 is cos 35°.

(b) Given cos B = .6561, to find $\angle B$.

(1) In which column is .6561?

(2) At which end of this column is cos found?

(3) Is .6561 the cosine of 41° or of 49°?

IV. Exercises in Reading from the Table of Functions

From the table, read the following functions:

1. sin 15°	**13.** tan 20°	**25.** sin 75°	**37.** cos 12°
2. cos 24°	**14.** cos 30°	**26.** cot 29°	**38.** sin 28°
3. tan 38°	**15.** sin 70°	**27.** cos 41°	**39.** cot 56°
4. cot 20°	**16.** cot 90°	**28.** tan 37°	**40.** tan 32°
5. cos 50°	**17.** cos 22°	**29.** cot 46°	**41.** sin 48°
6. tan 80°	**18.** tan 36°	**30.** sin 52°	**42.** cos 35°
7. sin 35°	**19.** cot 45°	**31.** cos 91°	**43.** cot 56°
8. cot 40°	**20.** sin 68°	**32.** tan 16°	**44.** tan 80°
9. tan 27°	**21.** tan 26°	**33.** cot 76°	**45.** cos 43°
10. sin 62°	**22.** cot 34°	**34.** tan 85°	**46.** cot 36°
11. cot 76°	**23.** sin 40°	**35.** cos 25°	**47.** sin 27°
12. cos 55°	**24.** cos 18°	**36.** sin 95°	**48.** tan 64°

Find the angle A, when,

49. sin A = .3584	**51.** cos A = .8572
50. tan A = .2679	**52.** cot A = 1.1918

53. $\sin A = .8090$	**57.** $\cot A = .5095$
54. $\cos A = .1736$	**58.** $\sin A = .7071$
55. $\tan A = 3.7321$	**59.** $\tan A = 1.0000$
56. $\cos A = .6428$	**60.** $\cos A = .4067$

61. (a) Compare $\sin 1°$ with $\tan 1°$;

$\sin 3°$ with $\tan 3°$;
$\sin 10°$ with $\tan 10°$;
$\sin 30°$ with $\tan 30°$;
$\sin 60°$ with $\tan 60°$.

(b) Show geometrically that the tangent of an angle is larger than its sine.

62. (a) At the center of a unit circle, draw an acute angle, A, and draw $\sin A$.

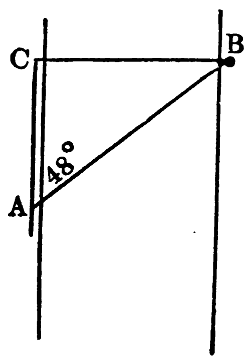
(b) By using the Pythagorean Theorem, show that $\sin^2 A + \cos^2 A = 1$.

63. From the equation, $\sin^2 A + \cos^2 A = 1$, show that

(a) $\sin A = \sqrt{1 - \cos^2 A} = \sqrt{(1 + \cos A)(1 - \cos A)}.$

(b) $\cos A = \sqrt{1 - \sin^2 A} = \sqrt{(1 + \sin A)(1 - \sin A)}.$

64. A boy, at a distance of 64 ft. from a school building, sighted to the top and found the angle of elevation to be 35°. Find the height of the building.



65. Some boys wished to measure the width of a busy street. From a point C opposite a telephone pole B on the other side, they measured a distance CA or 54 ft. along the sidewalk. The angle CAB was found to be 48°. What was the width of the street?

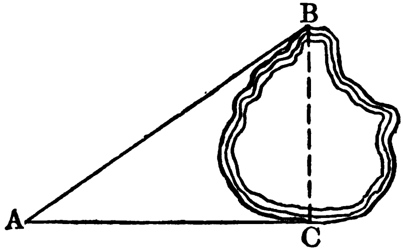
66. Draw a rectangle with a length of 6 in. Let the diagonal make an angle of 34° with the base. Calculate the width. Measure the width to see if it checks.

67. A girl placed a yardstick against a wall so that it made an angle of 67° with the floor. How far up the wall was the end of the stick placed? How far was the other end from the wall on the floor? Check your work by measuring the distances.

68. Draw an isosceles triangle with a base of 6 cm. and with base angles of 48°. Find the length of the sides and the altitude.

69. The gable end of a garage has a pitch of 42°. The width of the garage is 15 ft. How long are the rafters? What is the height of the gable?

70. A surveyor wishes to measure the distance across a pond. He sights across the pond to a tree at B. He surveys the line CA perpendicular to BC. He finds the distance AB to be 378 ft. and the angle CAB, 34°. What is the distance across the pond?



71. At a distance of 26 ft. from the foot of a telephone pole, some boys sight from the ground to the top of the pole and find the angle of elevation to be 46°. How high is the pole?

72. Some boy scouts stretched a heavy cord 24 ft. long from the top of a bluff, B, to a point, A, on the ground. The angle of elevation from point A to the top of the bluff was 52°. How high was the bluff?

CHAPTER TWELVE

NEW USES OF EXPONENTS

You have found it much easier and shorter to multiply and divide literal numbers than numerical numbers; for example, $x^4 \times x^7 = x^{11}$ and $x^{11} \div x^7 = x^4$. To multiply two like algebraic numbers, what must be done with their exponents? How does the process change in division? If x is 2, $x^4 = 16$ and $x^7 = 128$. To multiply 128 by 16 arithmetically is longer than to multiply x^4 by x^7. If we had a table showing different numbers as powers of 2, we could multiply the arithmetic numbers by adding their exponents as easily as literal numbers, for $2^4 \times 2^7 = 2^{11}$. Our table will show that $2^{11} = 2048$. Therefore, $16 \times 128 = 2048$.

A. CALCULATIONS BASED ON POWERS OF 2

1. By using the table, find the product of $1024 \times 32 \times 16$.

(a) *Solution:*

$$1024 = 2^{10}$$
$$32 = 2^5$$
$$16 = 2^4$$
$$1024 \times 32 \times 16 = 2^{10} \times 2^5 \times 2^4 = 2^{19}$$

But, from the table, $2^{19} = 524{,}288$

$$1024 \times 32 \times 16 = 524{,}288$$

(b) Find the product of 2048×64.

(c) Multiply 512 by 32.

(d) Find the product of $256 \times 128 \times 32$.

(e) Multiply 8192 by 128.

TABLE BASED ON 2

Number	Power of 2
2	2^1
4	2^2
8	2^3
16	2^4
32	2^5
64	2^6
128	2^7
256	2^8
512	2^9
1,024	2^{10}
2,048	2^{11}
4,096	2^{12}
8,192	2^{13}
16,384	2^{14}
32,768	2^{15}
65,536	2^{16}
131,072	2^{17}
262,144	2^{18}
524,288	2^{19}
1,048,576	2^{20}

2. By using the table, divide 262,144 by 1024.

(a) *Solution:*

$$262{,}144 = 2^{18}$$
$$1024 = 2^{10}$$
$$262{,}144 \div 1024 = 2^{18} \div 2^{10} = 2^8$$

But, from the table,

$$2^8 = 256$$

$$\therefore 262{,}144 \div 1024 = 256$$

3. Find the value of

$$\frac{8192 \times 65536 \times 512}{16384 \times 256 \times 128}$$

(a) *Solution:*

$$\frac{8192 \times 65536 \times 512}{16384 \times 256 \times 128} = \frac{2^{13} \times 2^{16} \times 2^9}{2^{14} \times 2^8 \times 2^7}$$
$$= 2^{(13+16+9-14-8-7)}$$
$$= 2^9$$
$$= 512$$

4. Multiply 256 by 512.

5. Divide 524,288 by 512.

6. Find the value of $\dfrac{1024 \times 32 \times 4096}{32{,}768 \times 512}$.

7. Divide 1,048,576 by 128.

8. Divide 262,144 by 4096.

9. Find the value of $131{,}072 \times 256 \div 2048$.

10. Find the value of $\dfrac{64 \times 256 \times 2048 \times 8192}{262{,}144 \times 512 \times 32 \times 2}$.

B. CALCULATIONS BASED ON POWERS OF 3

By using a table showing the powers of 3, we may make similar calculations with other numbers.

TABLE BASED ON 3

Numbers	Powers of 3
3	3^1
9	3^2
27	3^3
81	3^4
243	3^5
729	3^6
2,187	3^7
6,561	3^8
19,683	3^9
59,049	3^{10}
177,147	3^{11}
531,441	3^{12}
1,594,323	3^{13}
4,782,969	3^{14}
14,348,907	3^{15}

From the table based on the powers of 3 make the following calculations.

1. Multiply 2,187 by 27 by 243.
2. Divide 6,561 by 81.
3. Find the value of

$$729 \times 2{,}187 \times 3.$$

4. Find the value of

$$\frac{243 \times 19{,}683 \times 531{,}441}{27 \times 729 \times 2{,}187 \times 9}.$$

5. Divide 14,348,907 by 177,147.
6. Find the value of $4{,}782{,}969 \div 59{,}049 \times 81$.
7. Multiply 177,147 by 81.
8. Multiply 27 by 243 by 729.
9. Find value of $\dfrac{6{,}561 \times 243 \times 2{,}187}{729 \times 27 \times 81}$.
10. Divide 59,049 by 27.

C. ZERO AS AN EXPONENT

1. In dividing 2^3 by 2^3, we get 2^{3-3}, or 2^0, as the quotient. But any number divided by itself equals 1. Therefore $1 = 2^0$.

2. Show that $1 = 3^0 = 5^0 = 10^0 = x^0$.

3. What is the value of any number with an exponent of zero?

D. USING FRACTIONAL EXPONENTS

1. Tables showing the powers of 5, 6, 7, 10, etc., would assist in calculations with other numbers. But even many such tables would not contain all of the numbers. However, we can express all numbers as powers of 2, if we use *fractional exponents*.

(a) For example, since $8 = 2^3$ and $16 = 2^4$, every number between 8 and 16 may be expressed as 2 with an exponent of 3 +, (2^{3+}). Thus $15 = 2^{3.9}$. You see 15 is nearly 16, therefore the fractional exponent, 3.9, must be nearly 4.

(b) Similarly, 5 is between 4 and 8; therefore 5 is between 2^2 and 2^3. Based on the table of 2, $5 = 2^{2.3223}$.

(c) By using fractional exponents, all numbers can be expressed as powers of 3, 5, 7, 10, or of any other convenient number.

E. LOGARITHMS BASED ON 10

1. (a) In the seventeenth century, two great mathematicians, John Napier of Scotland and Henry Briggs of England, constructed tables to help make their calculations in astronomy easier and shorter.

John Napier

(b) They called their exponents *logarithms.* The last part of the word *logarithm,* comes from the Greek word *arithmos,* which means number. We get our word *arithmetic* from a form of *arithmos.* The first part of logarithm comes from the Greek *logos,* which means *word.* The verb form of *logos* means, to *speak* or to *tell.* These tables are aptly named tables of *logarithms* because *they tell the numbers,* without calculation.

2. Instead of using tables based on 2 or 3, as we have, Briggs made his logarithms based on 10. These are the

logarithms most commonly used. The abbreviation for logarithm is *log*.

TABLE BASED ON 10

Number	Power of 10
1	10^0
10	10^1
100	10^2
1,000	10^3
10,000	10^4
100,000	10^5
1,000,000	10^6

I. Integral Parts of Logarithms — Characteristics

1. Since

$$
\begin{array}{rll}
1 = 10^0, & \text{then, log} & 1 = 0 \\
10 = 10^1, & \text{then, log} & 10 = 1 \\
100 = 10^2, & \text{then, log} & 100 = 2 \\
1{,}000 = 10^3, & \text{then, log} & 1{,}000 = 3 \\
10{,}000 = 10^4, & \text{then, log} & 10{,}000 = 4 \\
100{,}000 = 10^5, & \text{then, log} & 100{,}000 = 5 \\
\text{etc.} & \text{etc.} &
\end{array}
$$

2. Any number between 1 and 10 has a logarithm of 0 plus a decimal. Any number between 10 and 100 has a logarithm of 1 plus a decimal, because its logarithm must lie between 1 and 2. For example, log 36 = 1.54407, because $36 = 10^{1.54407}$.

3. What must be the integral part of log 1250? of log 985? of 87624?

4. (a) The integral part of a logarithm is called its *characteristic*.

(b) The characteristic of the logarithm of an integer shows how many digits the number has. The decimal part of the logarithm of 25 is the same as that of 250. But the characteristic of 25 is 1, while that of 250 is 2.

Table of Logarithms

N	0	1	2	3	4	5	6	7	8	9
10	0000	0043	0086	0128	0170	0212	0253	0294	0334	0374
11	0414	0453	0492	0531	0569	0607	0645	0682	0719	0755
12	0792	0828	0864	0899	0934	0969	1004	1038	1072	1106
13	1139	1173	1206	1239	1271	1303	1335	1367	1399	1430
14	1461	1492	1523	1553	1584	1614	1644	1673	1703	1732
15	1761	1790	1818	1847	1875	1903	1931	1959	1987	2014
16	2041	2068	2095	2122	2148	2175	2201	2227	2253	2279
17	2304	2330	2355	2380	2405	2430	2455	2480	2504	2529
18	2553	2577	2601	2625	2648	2672	2695	2718	2742	2765
19	2788	2810	2833	2856	2878	2900	2923	2945	2967	2989
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133
26	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298
27	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456
28	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609
29	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757
30	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900
31	4914	4928	4942	4955	4969	4983	4997	5011	5024	5038
32	5051	5065	5079	5092	5105	5119	5132	5145	5159	5172
33	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302
34	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428
35	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551
36	5563	5575	5587	5599	5611	5623	5635	5647	5658	5670
37	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786
38	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899
39	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010
40	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117
41	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222
42	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325
43	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425
44	6435	6444	6454	6464	6474	6484	6493	6503	6513	6522
45	6532	6542	6551	6561	6571	6580	6590	6599	6609	6618
46	6628	6637	6646	6656	6665	6675	6684	6693	6702	6712
47	6721	6730	6739	6749	6758	6767	6776	6785	6794	6803
48	6812	6821	6830	6839	6848	6857	6866	6875	6884	6893
49	6902	6911	6920	6928	6937	6946	6955	6964	6972	6981
50	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067
51	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152
52	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235
53	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316
54	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396

Table of Logarithms (*Continued*)

N	0	1	2	3	4	5	6	7	8	9
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474
56	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551
57	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627
58	7634	7642	7649	7657	7664	7672	7679	7686	7694	7701
59	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774
60	7782	7789	7796	7803	7810	7818	7825	7832	7839	7846
61	7853	7860	7868	7875	7882	7889	7896	7903	7910	7917
62	7924	7931	7938	7945	7952	7959	7966	7973	7980	7987
63	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055
64	8062	8069	8075	8082	8089	8096	8102	8109	8116	8122
65	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189
66	8195	8202	8209	8215	8222	8228	8235	8241	8248	8254
67	8261	8267	8274	8280	8287	8293	8299	8306	8312	8319
68	8325	8331	8338	8344	8351	8357	8363	8370	8376	8382
69	8388	8395	8401	8407	8414	8420	8426	8432	8439	8445
70	8451	8457	8463	8470	8476	8482	8488	8494	8500	8506
71	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567
72	8573	8579	8585	8591	8597	8603	8609	8615	8621	8627
73	8633	8639	8645	8651	8657	8663	8669	8675	8681	8686
74	8692	8698	8704	8710	8716	8722	8727	8733	8739	8745
75	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802
76	8808	8814	8820	8825	8831	8837	8842	8848	8854	8859
77	8865	8871	8876	8882	8887	8893	8899	8904	8910	8915
78	8921	8927	8932	8938	8943	8949	8954	8960	8965	8971
79	8976	8982	8987	8993	8998	9004	9009	9015	9020	9025
80	9031	9036	9042	9047	9053	9058	9063	9069	9074	9079
81	9085	9090	9096	9101	9106	9112	9117	9122	9128	9133
82	9138	9143	9149	9154	9159	9165	9170	9175	9180	9186
83	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238
84	9243	9248	9253	9258	9263	9269	9274	9279	9284	9289
85	9294	9299	9304	9309	9315	9320	9325	9330	9335	9340
86	9345	9350	9355	9360	9365	9370	9375	9380	9385	9390
87	9395	9400	9405	9410	9415	9420	9425	9430	9435	9440
88	9445	9450	9455	9460	9465	9469	9474	9479	9484	9489
89	9494	9499	9504	9509	9513	9518	9523	9528	9533	9538
90	9542	9547	9552	9557	9562	9566	9571	9576	9581	9586
91	9590	9595	9600	9605	9609	9614	9619	9624	9628	9633
92	9638	9643	9647	9652	9657	9661	9666	9671	9675	9680
93	9685	9689	9694	9699	9703	9708	9713	9717	9722	9727
94	9731	9736	9741	9745	9750	9754	9759	9763	9768	9773
95	9777	9782	9786	9791	9795	9800	9805	9809	9814	9818
96	9823	9827	9832	9836	9841	9845	9850	9854	9859	9863
97	9868	9872	9877	9881	9886	9890	9894	9899	9903	9908
98	9912	9917	9921	9926	9930	9934	9939	9943	9948	9952
99	9956	9961	9965	9969	9974	9978	9983	9987	9991	9996

(c) In the table on page 191 you will see that the characteristic of the *logarithm* of any *integer* is *always* 1 *less than its number of digits.*

III. Decimal Parts of Logarithms — Mantissas

1. (a) The logarithms of all numbers, except perfect powers of 10, have *decimal parts added* to the *characteristic.* The *decimal part* of a logarithm is its *mantissa. Mantissa* is the Latin word which means *addition.* The mantissa is the part of the logarithm that is *added to* the characteristic.

(b) Tables of logarithms give only the *mantissas* of the *logarithms.* The characteristic is so readily found that it is usually omitted from the table.

IV. Explanation of Table of Logarithms

1. The given table is only a part of a complete set of logarithms and shows the logarithms of numbers from 10 to 99. Complete tables give a double page to each hundred numbers from 100 to 1000. This partial table will serve to illustrate the use of logarithms, but the student must realize that the results obtained will not be as accurate as those derived from the longer tables.

2. This table shows only four decimal places of the mantissas. Most tables give five, six, or seven places. It is readily understood that the ones with more places will give more accurate results.

3. Only *mantissas* are given in the table. The decimal points are not written, but must be supplied when used.

4. At the left of each page is a column of numbers. At the top of each is a row of the digits from 0 to 9. The column gives a number of two digits, as 10, which may be made one of three digits, as 105, by counting over to the column under 5.

5. In writing the logarithm of a number, first calculate its characteristic from the number of digits it has.

$$\log 105 = 2.+$$

6. In the number column, find the number made by the first two figures of 105, that is, the number 10. *Opposite* 10 and *under* 5 is the mantissa for 105, which is .0212. This decimal must be added to the characteristic already found.

Therefore, log 105 = 2.0212.

7. Find the logarithm of 772.
   (a) What is the characteristic of log 772?
   (b) Opposite what number and under which digit will you find the mantissa?
   (c) Do you find log 772 = 2.8876?

V. Finding Logarithms of Numbers

Find the logarithms of the following numbers.

	(a)	(b)	(c)	(d)
1.	55	558	5580	5
2.	69	690	699	6990
3.	6	80	808	8080
4.	72	725	720	7250
5.	36	369	3690	306
6.	8	86	867	8670
7.	47	479	749	7490
8.	9	93	903	934

VI. Finding Numbers from Logarithms

1. If one knows the logarithm of a number, the number itself may be found from the table.

2. The *number corresponding* to a given logarithm is an *antilogarithm.*
   (a) The prefix *anti-* means *opposite to* or *corresponding to.*
   (b) *Antilogarithm* is abbreviated *antilog.*

3. Find the number that corresponds to the logarithm 3.7832. The mantissa 7832 is in the row opposite 60 and in the column under 7. Therefore, the first three figures of the number are 607. The characteristic, 3, is 1 less than the number of digits. Therefore, the number must have 4 digits. Zero must be annexed to make the required periods.

$$\therefore \text{antilog } 3.7832 = 6070.$$

This equation means that 6070 is the number whose logarithm is 3.7832; for, $6070 = 10^{3.7832}$.

4. Find the antilogarithms of the following:

(a) 0.0414	(f) 0.9294	(k) 2.8887
(b) 4.9542	(g) 2.9165	(l) 3.6107
(c) 2.1818	(h) 1.8325	(m) 0.5563
(d) 0.3979	(i) 2.3766	(n) 1.2788
(e) 3.7210	(j) 3.6955	(o) 5.9430

VII. Multiplication by Logarithms

1. Since logarithms are exponents, we may multiply two or more numbers by adding their logarithms. The sum will be the logarithm of the product. The antilogarithm gives the numerical product.

2. Multiply 17 by 24.

Solution:

$$\begin{array}{r l}
\log 17 = & 1.2304 \\
\log 24 = & 1.3802 \\
\hline
\log (17 \times 24) = & 2.6106 \text{ (adding logs)} \\
\text{antilog } 2.6106 = & 408
\end{array}$$

(a) This table gives $\log 408 = 2.6107$. A five-place table gives $\log 408 = 2.61066$. With the larger table, the product is found exactly.

$$\begin{array}{r l}
\log 17 = & 1.23045 \\
\log 24 = & 1.38021 \\
\hline
\log (17 \times 24) = & 2.61066 \\
\text{antilog } 2.61066 = & 408
\end{array}$$

3. (a) The product of three or more numbers may be found directly by adding their logarithms.

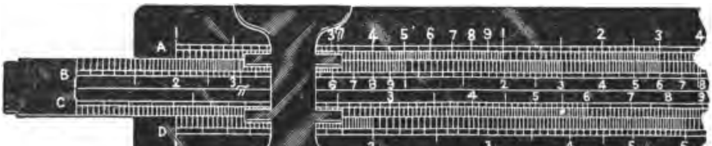
(b) Find the product of $12 \times 17 \times 19$ by using logarithms.

Solution:

Four-place Table	Five-place Table
$\log 12 = 1.0792$	$\log 12 = 1.07918$
$\log 17 = 1.2304$	$\log 17 = 1.23045$
$\log 19 = 1.2788$	$\log 19 = 1.27875$
$\log 12 \times 17 \times 19 = 3.5884$	$\log 12 \times 17 \times 19 = 3.58838$
antilog $3.5884 = 3880$	antilog $3.58838 = 3876$
$\therefore 12 \cdot 17 \cdot 19 = 3880$ approximately	$\therefore 12 \cdot 17 \cdot 19 = 3876$ exactly

4. In your later work in algebra and in trigonometry, you will learn to use the more complete tables which give the more exact results. From the four-place tables, you may learn the meaning and use of logarithms, but you must not think that they are impractical because these results are only approximate. Scientists, draftsmen, and engineers use logarithms for nearly all of their calculations, because they are such a labor-saving device.

An instrument has been invented, known as the *slide-rule*, by which many calculations may be made mechanically.



The principle underlying the working of the slide-rule is that of logarithms.

5. $35 \times 8 \times 9$

6. $650 \times 48 \times 15$

7. $20 \times 8 \times 23$

8. $16 \times 11 \times 30$

9. $13 \times 40 \times 120$

10. $720 \times 80 \times 50$

VIII. Division by Logarithms

1. To divide x^{10} by x^3, what must be done with the exponents?

2. Divide 2^9 by 2^6.

3. Divide $10^{1.5911}$ by $10^{1.1139}$.

Division of numbers expressed with fractional exponents is done by subtracting their exponents, in the same way as with integral exponents. The same reasoning holds for logarithms, because logarithms are exponents.

4. Divide 3900 by 130, using logarithms.

$$\begin{aligned} \log 3900 &= 3.5911 \text{ or } 3.59106 \\ \underline{\log\ \ 130} &= \underline{2.1139} \text{ or } \underline{2.11394} \\ \log (3900 \div 130) &= 1.4772 \text{ or } 1.47712 \text{ (subtracting logs)} \end{aligned}$$

antilog 1.4772 = 30 approximately.
antilog 1.47712 = 30 exactly.

5. (a) Divide by logarithms.
(b) Check by arithmetic division:

(1) $891 \div 33$	(4) $6930 \div 21$
(2) $832 \div 26$	(5) $7980 \div 133$
(3) $776 \div 97$	(6) $94800 \div 790$

6. (a) Perform the indicated operations by logarithms.
(b) Check by arithmetic calculations.

(1) $\dfrac{4960 \times 525}{744}$

(2) $\dfrac{9920 \times 675}{558}$

(3) $\dfrac{434 \times 840}{588}$

(4) $\dfrac{594 \times 328}{902}$

CHAPTER THIRTEEN

TEST OF ACHIEVEMENT

A. PROBLEMS

1. A man receives a salary check of \$150, pays \$30 rent, \$2 gas bill, \$2 light bill, \$2 telephone bill, receives \$5, spends \$30 for clothes, receives a check of \$10, pays a grocery bill of \$15, buys a piece of furniture for \$75, and pays \$3 for books and magazines. How much has he left?

Solve this problem in two ways.

2. A boy has \$50 in the bank and \$5 cash. How much is he worth if he owes \$14 for a coat and \$8 for shoes?

3. A boy put \$15 in the bank, wrote a check for \$2, deposited \$8 and \$20, wrote a check for \$6 and another for \$25. How much had he left in the bank?

4. Find the following sums both horizontally and vertically:

$$\begin{array}{r}
6+2-3-8+6-9-7+3-4=\\
-7+3-8+9-2+6+4-2+6=\\
+3-1+7-6+4-3-7+8-6=\\
8+2-6-7+2+10-3-3-6=\\
\underline{1-6+7+3+5-4+6+9+3=}
\end{array}$$

5. Draw any unmeasured line n inches long.

Draw a rectangle $4n + 3$ inches long and $3n - 2$ inches wide. Find the perimeter.

6. (a) Find the perimeter of each rectangle from the following dimensions.

(b) Give a measure to each line and find the numerical length of each perimeter.

No.	Length	Width	Perimeter	Numerical Lengths	
				Of Line	Of Perimeter
(1)	$x + 6$	$x - 7$	....	......	....
(2)	$n + 5$	$n - 2$	....	......	....
(3)	$a + 11$	$a - 8$	....	......	....
(4)	$y + 7\frac{1}{2}$	$y - 5\frac{1}{2}$	....	......	....
(5)	$b + 9$	$b - 6$	....	......	....
(6)	$z + 3.2$	$z - 1.6$	....	......	....
(7)	$c + 8$	$c - 3$	....	......	....
(8)	$m + 10$	$m - 4$	....	......	....
(9)	$t + 7$	$t - 5$	....	......	....
(10)	$r + 1$	$r - 1$	....	......	....

7. Draw two unmeasured lines. Let the longer one be x inches and the shorter one be y inches long.

Draw a rectangle $2x + 3y$ inches long and $3x - 2y$ inches wide. Find the perimeter.

8. Draw a rectangle $6x - y$ inches long and $4x - 3y$ inches wide. Find the perimeter.

9. (a) Draw rectangles having the dimensions given below.

(b) Find the perimeter and area of each.

(c) Find the numerical value of the perimeter and area of each, using the given lengths for the lines.

No.	Length	Width	Perimeter	Area	Numerical Value of		
					Line	Perimeter	Area
(1)	$x + 8$	$x - 7$	......	......	$x = 9$	......	......
(2)	$n + 5$	$n - 3$	......	......	$n = 7$	......	......
(3)	$3a - 4$	$a - 2$	......	......	$a = 5$	......	......
(4)	$5y + 7$	$2y - 1$	......	......	$y = 3$	......	......
(5)	$4b + .5$	$3b - .2$	......	......	$b = 1$	......	......
(6)	$3z - 8$	$z + 3$	......	......	$z = 7$	......	......
(7)	$9c + 5$	$9c - 1$	......	......	$c = \frac{1}{3}$	......	......
(8)	$4r + 7$	$16r - 5$	......	......	$r = \frac{1}{2}$	......	......
(9)	$\frac{1}{2}x + 6$	$\frac{1}{2}x - 1$	......	......	$x = 8$	......	......
(10)	$7n - 4$	$6n - 5$	......	......	$n = 2$	......	......

(d) Make up and solve 5 more problems like the above.

10. (a) Without pencil, find the following products.
(b) Explain the short cuts used.

(1) 27×33	(8) 18×16	(15) 20×12	(22) 21^2
(2) 36×38	(9) 19×15	(16) 13×11	(23) 31^2
(3) 37×43	(10) 20×14	(17) 14×12	(24) 41^2
(4) 19×17	(11) 21×13	(18) 15×11	(25) $(5\frac{1}{2})^2$
(5) 20×16	(12) 17×15	(19) 15×13	(26) $(6\frac{1}{2})^2$
(6) 21×15	(13) 18×14	(20) 16×12	(27) $(7\frac{1}{2})^2$
(7) 22×14	(14) 19×13	(21) 11×17	(28) 45^2

11. Without pencil, find the products of the following expressions:

(a) $(b - 2c)(b - 2c)$	(k) $(x - 7)(x - 7)$
(b) $(x + 8z)(x + 8z)$	(l) $(3a + 5)(3a + 5)$
(c) $(3a - 7x)(2a + x)$	(m) $(9m - y)(3m + 2y)$
(d) $(6x + 5y)(6x - 5y)$	(n) $(12c - d)(12c + d)$
(e) $(7a + 2r)(3a - 8r)$	(o) $(5x + 3)(2x - 7)$
(f) $(b - 3x)(b - 3x)$	(p) $(3b - 6)(3b - 6)$
(g) $(x + 9)(x + 9)$	(q) $(7x + 8)(7x + 8)$
(h) $(5c - 4)(8c + 7)$	(r) $(4c - 5b)(2c + 7b)$
(i) $(8x - 3y)(8x + 3y)$	(s) $(6d + 7e)(6d - 7e)$
(j) $(4a + b)(6a - 5b)$	(t) $(3a + 4c)(5a - 2c)$

12. Find the factors of the following:

(a) $2x^2 + 7xy + 6y^2$	(j) $4b^2 - 9x^2$
(b) $2x^2 - 7xy + 6y^2$	(k) $10x^2 + 41xy + 21y^2$
(c) $2x^2 + xy - 6y^2$	(l) $12x^2 - 35ab + 18b^2$
(d) $2x^2 - xy - 6y^2$	(m) $21c^2 + cd - 10d^2$
(e) $x^2 - y^2$	(n) $3v^2 - 5vt - 2t^2$
(f) $8a^2 + 14a + 3$	(o) $49b^2 - 16x^2$
(g) $15x^2 - 28xy + 12y^2$	(p) $16c^2 + 42cd + 5d^2$
(h) $24c^2 + 22cd - 7d^2$	(q) $10a^2 - 27ab + 18b^2$
(i) $8v^2 - 6vt - 9t^2$	(r) $14b^2 + bx - 3x^2$

13. Check the factors of each by giving convenient numerical values to the letters in each problem in Exercise 12.

14. (a) Find the area of the base and the volume of each cube or oblong from the given dimensions.
(b) From the numerical values given to the letters, find the numerical measure of each volume.
(c) Check each.

No.	Length	Width	Height	Area of Base	Volume	Numerical Value of Lines	Numerical Value of Volume
(1)	$a + 6$	$a + 5$	$a + 2$	.....	.....	$a = 3$	.....
(2)	$2x + 3$	$2x + 3$	$2x + 3$	.....	.....	$x = 5$	.....
(3)	$5a + 6$	$3a - 4$	$3a + 2$	.....	.....	$a = 6$	.....
(4)	$3y + 2z$	$2y + z$	$y - \frac{1}{2}z$	.....	.....	$y = 4, z = 2$	.....
(5)	$7a + 2b$	$5a - 3b$	$3a + b$	.....	.....	$a = 5, b = 1$	.....
(6)	$5c - 6d$	$4c - 9d$	$2c + 3d$	.....	.....	$c = 2, d = \frac{1}{3}$	.....
(7)	$12x + 7y$	$8x - 3y$	$20x - y$	.....	.....	$x = \frac{1}{2}, y = 3$	.....
(8)	$16m - n$	$10m - n$	$6m - n$	.....	.....	$m = \frac{1}{2}, n = 2$	.....
(9)	$4b + 7x$	$2b + 9x$	$3b - x$	.....	.....	$b = 5, x = 2$	.....
(10)	$3a + 5b$	$3a + 5b$	$3a + 5b$	.....	.....	$a = 6, b = .2$	.....
(11)	$7c + 8x$	$4c + 3x$	$5c - 2x$	.....	.....	$c = 3, x = 4$	.....
(12)	$4d + 7y$	$2d + 9y$	$8d - 3y$	.....	.....	$d = 3, y = 2$	.....
(13)	$4x - .5y$	$3x - .2y$	$3x - y$	.....	.....	$x = 5, y = 10$	.....
(14)	$6r + 5s$	$21r - s$	$15r - 2s$	.....	.....	$r = \frac{1}{3}, s = 1$	.....
(15)	$4x - 9$	$4x - 9$	$4x - 9$	.....	.....	$x = 9$	.....
(16)	$4d - x$	$5d - 2x$	$3d - x$	.....	.....	$d = 4, x = 6$	.....
(17)	$2a + 7b$	$5a + 3b$	$6a + b$	.....	.....	$a = 3, b = 3$	.....
(18)	$3h + 2k$	$3h + 2k$	$5h - 6k$	.....	.....	$h = 7, k = 2$	.....
(19)	$20x + 3y$	$10x + 7y$	$10x + 7y$	.....	.....	$x = .3, y = .5$	.....
(20)	$8a - 7$	$8a - 7$	$8a - 7$	.....	.....	$a = 4$	.....

15. Use or make a formula for the solution of each of the following problems.
(a) How long will it take a train to run 540 miles, if the rate is 40 miles an hour?
(b) If a boy solves 4 problems a minute how long will it take him to solve 100 problems?
(c) If a boy saves 2 dollars a week how long will it take him to save 50 dollars?
(d) The length of a rabbit pen is three times its width. If the perimeter of the pen is 48 ft., find the length and width.
(e) It takes 1386 sq. ft. of sod to cover a circular

grass plot. How long must a walk be which extends from the edge of the grass plot to the center where a bench is placed? (πr^2 = Area.)

(f) How long will it take 250 dollars to produce 42 dollars interest at the rate of 4%?

(g) Make a problem and a formula for it about cost; about the radius of circle in terms of circumference.

16. The length of a field is 8 rods more than twice its width. If the perimeter is 256 rods, find the two dimensions.

17. The length of a book is $2\frac{1}{2}$ inches more than the width. If the perimeter is 25 inches, find the dimensions.

18. The length of a kitchen table is 12 inches more than the width. If the area is 1120 sq. in., what will be the length and width of oilcloth for the table? (Neglect the amount needed for turning over edges.)

(a) Solve the equation several ways including graphs.

19. A girl is 2 years more than twice as old as her brother. The product of their ages is 84. How old is each?

20. Three boys solve 35 problems in a study period. The second boy solves twice as many problems as the first, and the third solves four times as many as the first. How many does each solve?

21. Six times a number decreased by 8 equals eight times the number decreased by 24. What is the number?

22. 52 exceeds 4 times a number by as much as the number exceeds 8. Find the number.

23. A boy raised $6\frac{3}{4}$ bushels of vegetables in his garden. If he raised $\frac{5}{8}$ as many bushels of beans as potatoes, and $1\frac{3}{4}$ times as many bushels of tomatoes as potatoes, how many bushels of each kind did he raise?

24. A suit of clothes sold for $50, which was at a gain of $33\frac{1}{3}$% above cost. What was the cost?

25. A girl's dress was sold for $16.50 after a discount of

25% was taken from the marked price. What was the marked price?

26. A boy weighing 110 lb. is 6 ft. from the fulcrum of a teeter board. How far from the fulcrum must the boy on the other end be, if he weighs 96 lb.?

27. Two girls bought some fruit. One girl bought 2 apples and 3 bananas and paid 26 cents. The other bought 4 apples and 5 bananas for 46 cents. Find the cost of each. (Solve both algebraically and graphically.)

28. (a) Solve the following sets of equations algebraically.

(b) Solve each set graphically.

(1) $5x - 2y = 21$
$2x + y = 12$

(2) $x + y = 4$
$4x + 5y = 23$

(3) $3x - 2y = 4$
$6x - 5y = 16$

(4) $7x - 2y = 11$
$4x + y = 7$

(5) $6y - x = 5$
$4y - x = 3$

(6) $3a - 5b = 1$
$5a - 3b = 23$

(7) $4x + y = 6$
$8x + 2y = 12$

(8) $3x + 5y = 31$
$y = \dfrac{29 - 3x}{5}$

29. How long will it take a bomb to fall from an airplane 4800 ft. above the earth? ($S = \frac{1}{2}gt^2$; g is approximately 16.)

30. The product of two consecutive numbers is 306. Find the numbers.

31. The sum of two numbers is 8. The product is 10. Find the numbers. (For square roots of small numbers, use the table given in the Appendix.)

32. Simplify the following problems:

(a) $\dfrac{x+2}{x^2-5x-14} + \dfrac{1}{x^2-4x-21} - \dfrac{x-7}{x+3}.$

(b) $\dfrac{3}{a+x} - \dfrac{2}{a-x} - \dfrac{5x}{a^2+2ax+x^2}$

(c) $\dfrac{3x + 2y}{7} - \dfrac{5x - 6y}{42} + \dfrac{8x - 7y}{6}$

(d) $\dfrac{7}{a + b} - \dfrac{6a}{a^2 - 2ab + b^2} + \dfrac{9b}{a^2 - b^2}$

(e) $\dfrac{4c + 7d}{5c + 9d} - \dfrac{4c^2 + 77cd + 21d^2}{5c^2 - 6cd - 27d^2} + \dfrac{4c}{c - 3d}$

(f) $\dfrac{3s^2 - 5st - 8t^2}{s^2 - 5st - 6t^2} \div \dfrac{3s - 8t}{s^2 - 7st + 6t^2} \div \dfrac{s - t}{s}$

33. The base of a right triangle is 3.5 in. longer than the altitude. The hypotenuse is 6.5 in. Find the base and altitude.

34. The base of a right triangle is 1.75 in. longer than the altitude. The hypotenuse is 3.25 in. Find the base and altitude.

35. A boy standing 100 ft. from the foot of a building finds the angle of elevation of the top to be 32°. If his eye is 5 ft. from the ground, how high is the building?

36. Some boys stretched 12 ft. of steel tape from the top of a porch to the ground. The angle of elevation from the ground to the porch was 41°. How high was the porch?

37. Draw an isosceles triangle with a base of 6 in. and an altitude of 3 in. With your protractor, measure an angle of the triangle. From this, calculate the length of the legs of the triangle.

38. Draw a right triangle with a base angle of 48° and a hypotenuse of 2.5 in. Calculate the altitude of the triangle.

39. Draw a line x cm. long. At one end of it, draw a perpendicular 5 cm. long. Join the top of the perpendicular to the other end of line x. Measure one of the acute angles of the triangle. Calculate the length of line x. Find the length of the hypotenuse.

40. Solve by logarithms

$$\frac{5760 \times 7560.}{540 \times 1920}$$

CHAPTER FOURTEEN

GEOMETRY

A. EARLY MATHEMATICIANS

The meaning of the word *geometry* is *measure* of the *earth,* which shows that, in earliest times, geometry dealt with surveying or measurement of land. The beginnings of geometry may be traced back to the Egyptians and Phœnicians. It had a very practical use among the Egyptians. It was the Greeks, however, who did most in developing the subject.

I. Thales

Thales was the founder of the earliest Greek school of mathematics. When did he live? Tell the story of his mule and sponges, how he "cornered" the olive market, and why he sacrificed an ox to the immortal gods. For his time, Thales was a skilled geometer, although he probably knew no more than six theorems. (See Book Two.)

Thales

II. Pythagoras

About the time Thales died, Pythagoras was born. He learned geometry from some of the followers of Thales and later founded schools and a secret brotherhood. His fol-

lowers were known as Pythagoreans. He knew a great deal more than Thales about parallels, triangles, and parallelograms. What particular theorem about the right triangle bears his name? Pythagoras was the first to arrange the leading propositions, or theorems, in a logical order, a sort of chain of truths, each linked to the one preceding.

III. Hippocrates of Chios

The first elementary textbook of geometry was written by Hippocrates of Chios, who was born in 470 B.C., about thirty years after the death of Pythagoras. He denoted the square on a line by the Greek word, *dynamis*, which means *power*. Show the meaning of its English derivatives, *dynamite*, *dynamo*, and *dynamic*. We use power in the same meaning when we speak of the *second power* of a or a^2. We have extended its use to *higher powers*, as a^3 or a^4.

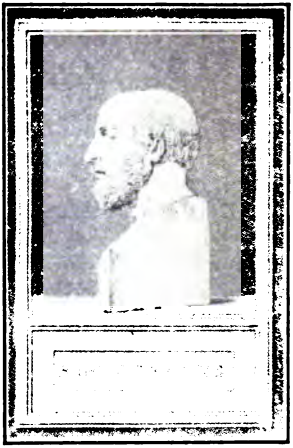
IV. Plato

Another famous teacher of geometry was Plato, who was born at Athens in 429 B.C. Above the door of his school was the inscription, "Let none ignorant of geometry enter my door." It was Plato who decreed that in the constructions of elementary geometry, only the compass and the unmarked straight edge, or ruler, should be used.

B. THE MOST FAMOUS BOOK ON GEOMETRY AND ITS AUTHOR

About three hundred years after the birth of Thales, was born the man who wrote a book which has had more editions than any other book except the Bible. The man's name was *Euclid*. His book is known as *Euclid's Elements*. Translations of his book are now used for the study of geometry in schools of all countries. A school boy in England or in Canada does not speak of studying geometry.

Pythagoras



Hippocrates

Plato

Euclid

He says, "I am studying Euclid." The geometry texts used in American schools are based upon Euclid, and follow his order and general plan, but they are abridged and modernized. In the three centuries between the times of Thales and Euclid, all the truths of elementary geometry were discovered. Euclid's Elements contained all that is given in modern texts and also some more advanced work.

C. WHAT GEOMETRY IS

Heretofore, your work in mathematics has been solving problems or making calculations, either with arithmetic numbers or algebraic letters. All problems in mensuration may be called geometric problems because they deal with form and measurement. For example, you have calculated the hypotenuse of a right triangle from the measures of the two legs, using the geometric truth called the Pythagorean Theorem to tell you how. In geometry you must prove by reasoning that the Pythagorean Theorem is true for all right triangles. Formal geometry is simply reasoning.

D. THE ART OF REASONING

I. Basis of Correct Reasoning

In an argument or debate, the speaker must start with a statement whose truth is accepted by his opponent. Then he must present or establish the truth of a second statement that is correctly related to the first. The two statements, called *premises*, are so related that if the opponent grants the truth of each one, he must admit the truth of the conclusion. For example, if one admits the truth of the statements, (1) "Good exercise is health-giving," and (2) "Playing ball is good exercise," he must admit the truth of the conclusion, which necessarily follows, namely, "Playing ball is health-giving."

The debater, therefore, does not concern himself about the conclusion which he wishes to draw, but tries to establish the truth of his two premises.

For convenience, the two premises are named *major* and *minor premises*, respectively.

II. Drawing Conclusions

From the following premises, draw the necessary conclusions.

1. (a) Major Premise: All flowers are beautiful.
   (b) Minor Premise: The rose is a flower.
   (c) Conclusion: Therefore, ————————.

2. (a) Major Premise: Good books are much read.
   (b) Minor Premise: Kipling's books are good.
   (c) Conclusion: Therefore, ————————.

3. (a) Major Premise: The base angles of an isosceles triangle are equal.
   (b) Minor Premise: Angles A and B are base angles of the isosceles triangle ABC.
   (c) Conclusion: Therefore, ————————

III. False Reasoning

Unless the two premises are always true and are correctly related, false conclusions may be drawn.

From the two premises, "Only contented people are wise" and "A tramp is contented in his rags," is it correct to conclude that "A tramp is a wise man"?

Are the following conclusions correct? If not, why?

1. To play all day is proof of great idleness. This violinist plays all day. Therefore, the violinist is a very idle person.

2. Warm days are sunny. This is a warm day. Therefore, it is sunny.

3. Base angles of an isosceles triangle are equal. A and B are the angles at the base of the triangle ABC. Therefore, angle A equals angle B.

IV. Abridged Forms in Reasoning

Usually in debates and arguments, the two premises and conclusion are not stated separately. Often the conclusion is stated first, followed by one premise introduced by *because* or *for* and the other premise is omitted but implied.

If one says "Kipling's books are much read, for they are good," what premise is omitted?

Write the following in the form of two premises and a conclusion.

1. This is a bracing morning, for it is clear and frosty.
2. Angle A equals angle B, for they are both right angles.
3. In the parallelogram $ABCD$, AB equals CD, for they are opposite sides of a parallelogram.

E. REASONING IN GEOMETRY

Because the forms of mensuration, lines, angles, triangles, circles, and other figures, are definite and easily represented in diagrams, they make the best and easiest topics with which to learn the art of reasoning. And because Euclid wrote in such a beautifully clear and logical style, his *Elements* have been used for over two thousand years. In the senior high school, you will learn more of his work. A few propositions, however, are given here, in order to show what the study of geometry is.

F. THE BASIS OF GEOMETRIC REASONING

Just as two debaters must begin by stating the facts and principles upon which they agree and must start their arguments from this common ground, so in geometry a few

facts and axioms are assumed. Most of these assumptions you have already made in your study of mensuration.

Make a list of these assumptions by answering the following questions.

1. What is the addition axiom? State the subtraction, multiplication, division, and equality axioms.

2. How many straight lines can be drawn between two points?

3. In how many points can two straight lines intersect?

4. Are all right angles equal?

5. Are all straight angles equal?

6. If two straight lines intersect, which of the angles formed are equal?

7. How many perpendiculars can be drawn to a line at a given point in the line?

8. How many perpendiculars can be drawn to a line from a given point outside the line?

9. Through a given point how many lines can be drawn parallel to a given line?

10. How do two complements of the same angle, or of equal angles, compare?

11. Compare two supplements of the same angle, or of equal angles.

12. How do the radii of the same circle, or of equal circles, compare?

13. Which parts of congruent figures are equal?

Other assumptions may be added to the list as needed.

G. HOW TO PROVE TWO TRIANGLES ARE CONGRUENT

I. Constructions

1. Draw a triangle ABC.

NOTE. — Be careful not to draw a special triangle, except when so directed. A "triangle ABC" should not be a right triangle. Its three sides should be unequal and its three angles unequal.

2. Construct three other triangles as follows:

(a) Draw $\triangle DEF$, making DE greater than AB, $\angle D$ greater than $\angle A$, and $DF = AC$.

(The symbol for *is greater than* is $>$; as, $\angle D > \angle A$.)

(b) Draw $\triangle MNO$, making $MN = AB$, $\angle M$ less than $\angle A$, and $MO = AC$.

(The symbol for *is less than* is $<$; as, $\angle M < \angle A$.)

(c) Draw $\triangle XYZ$, making $XY = AB$, $\angle X = \angle A$, and $XZ = AC$.

II. Superposing Triangles

1. (a) Cut out $\triangle ABC$ and place it upon $\triangle DEF$, that is, superpose it on $\triangle DEF$, by putting the vertex A on the vertex D.

(b) Do any other parts of the two triangles coincide?

(c) Keeping A exactly on D, adjust $\triangle ABC$, so that AB takes the direction of DE, that is, AB falls along DE.

(d) Why does point B fall on point E?

(e) Does AC take the direction of DF? Why?

(f) Since $\triangle ABC$ and DEF cannot be made to coincide, they are not congruent.

2. (a) Superpose $\triangle ABC$ on $\triangle MNO$, by placing point A on point M and letting AB take the direction of MN.

(b) Does point N fall on point B? Why?

(c) Does AC take the direction of MO? Why?

(d) Can $\triangle ABC$ be made to coincide with $\triangle MNO$?

(e) Why are these triangles not congruent?

3. (a) Superpose $\triangle ABC$ on $\triangle XYZ$, by placing point A on X and letting AB take the direction of XY, with points C and Z on the same side of XY.

(b) Does point A fall on point Y? Why?

(c) Does AB take the direction of XZ? Why?

(d) Does point C fall on point Z? Why?

(e) The two lines BC and YZ are now connecting the

same two points. How many straight lines may be drawn between two points?

(f) How much of the two triangles coincide?

(g) Why is $\triangle ABC$ congruent to $\triangle XYZ$?
(The symbol for *is congruent to* is $\cong$.)

(h) How many parts of $\triangle ABC$ were given equal to the corresponding parts of $\triangle XYZ$? Which parts were these?

III. Theorem on Congruence of Triangles

1. Are any two triangles congruent which have two sides and the included angle of one equal respectively to two sides and the included angle of the other? Reply to this question by changing it into a statement.

2. This statement is the first *theorem* in geometry on the congruence of triangles.

3. Every theorem has two parts:

(a) One, called the *hypothesis*, states what is given.

(1) Which part of this theorem is the hypothesis?

(b) The other part, the *conclusion*, states what is to be proved.

(1) Which part of this theorem is the conclusion?

4. From the hypothesis a figure may be drawn that illustrates its conditions. To prove the truth of the theorem, it is necessary to reason step by step from the hypothesis to the conclusion, using as authority the definitions, assumptions, and axioms. After a theorem has been proved, it may be used as authority in proving other theorems.

5. By answering the questions given in II, 3, you have demonstrated the first proposition on congruent triangles.

6. Write the theorem, draw the figure, state what is given in the hypothesis about the parts of the figure, state what the conclusion says must be proved about the figure, then write out each step of the proof by answering the questions given in II, 3, giving a reason for each statement. After it is written compare with the form of proof on the next page.

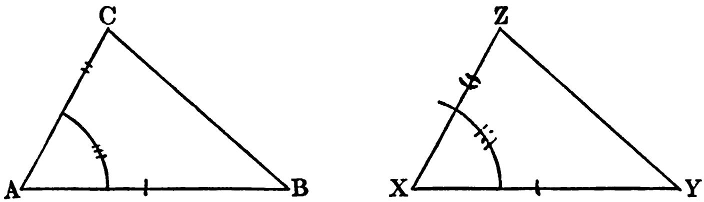
IV. Demonstrating the Truth of the Theorem

Proposition I

CONGRUENCE OF TRIANGLES

Theorem: Two sides and the included angle.

Two triangles are congruent if two sides and the included angle of one are equal respectively to two sides and the included angle of the other.



Given the triangles ABC and XYZ, with AB equal to XY, AC equal to XZ, and angle A equal to angle X.

To prove that $\triangle ABC \cong \triangle XYZ$.

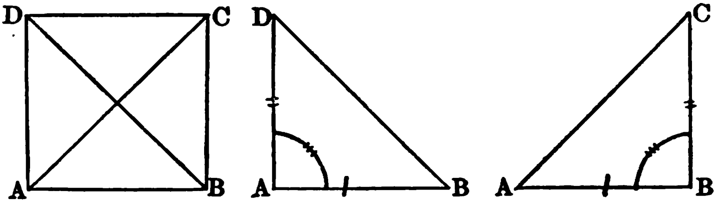
Proof. Method, superposition.

Arguments	Reasons
Place $\triangle ABC$ on $\triangle XYZ$, so that A falls on X, and AB falls along XY, and C and Z lie on the same side of XY.	
1. Then B will fall on Y,	1. For AB is given $= XY$.
2. AC will fall along XZ,	2. For $\angle A$ is given $= \angle X$.
3. and point C will fall on point Z,	3. For AC is given $= XZ$.
4. Then BC must coincide with YZ,	4. Only one straight line can be drawn between two points.
5. $\therefore \triangle ABC \cong \triangle XYZ$.	5. Two figures that coincide throughout are congruent.

V. Using the Theorem in Other Proofs

After proving this theorem, it is no longer necessary to superpose two triangles to prove that they are congruent. If you can show that these three parts of one triangle are equal respectively to the three corresponding parts of the other, you may claim that the two triangles are congruent and quote this theorem as authority.

1. For example, prove that the diagonals of a square are equal.

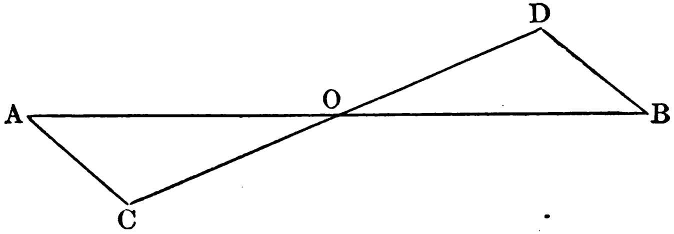


Given the square $ABCD$, with its diagonals, BD and AC.
To prove that $BD = AC$.
Proof. Method, congruent $\triangle$.

Arguments	Reasons
[Select a $\triangle$ that has AC as a side, ($\triangle ABC$). Select another that has BD as a side, ($\triangle ABD$). For convenience, these $\triangle$ may be drawn separately.]	
In the $\triangle$ ABD and ABC,	
1. $AB = AB$,	1. The same line, common to both triangles.
2. $AD = BC$.	2. The sides of a square are equal. (Def. of square.)
3. $\angle BAD = \angle ABC$,	3. The $\angle$ of a square are right $\angle$ and equal. (Def. of square.)
4. $\therefore \triangle ABD \cong \triangle ABC$,	4. Having two sides and included $\angle$ respectively equal.
5. $\therefore BD = AC$.	5. Homologous parts of $\cong$ figures are equal.

2. Prove that two right triangles are congruent if the two legs of one are equal respectively to the two legs of the other.

3. Draw two straight lines AB and CD bisecting each other at O. (Why should you not draw them perpendicular to each other?) Draw AC and BD.



(a) Prove $\triangle AOC = \triangle BOD$.
(b) Prove $AC = BD$.
(c) Prove $\angle A = \angle B$.
(d) Prove $\angle C = \angle D$.

4. Bisect the base of a square. From the point of bisection, draw lines to the opposite vertices.
(a) Prove that these lines are equal.
(b) Prove that they make equal angles with the base at the point of bisection.

5. (a) Draw a rectangle 8 cm. by 6 cm.
(b) Draw a diagonal.
(c) Prove that the diagonal divides the rectangle into two congruent triangles.
(d) Select two pairs of equal angles.
(e) Draw the second diagonal.
(f) Prove the two diagonals are equal.
(g) Compute the length of each diagonal.
(h) Select eight other pairs of equal angles.

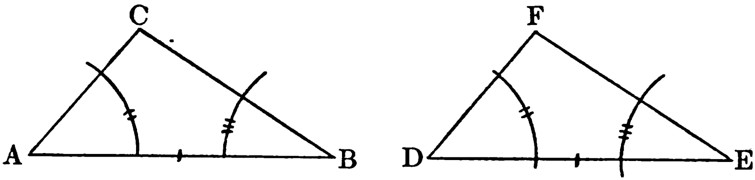
VI. Demonstration

Proposition II

CONGRUENCE OF TRIANGLES

Theorem: Two angles and included side.

Two triangles are congruent if two angles and the included side of one are equal respectively to two angles and the included side of the other.



Given the triangles ABC and DEF, with AB equal to DE, angle A equal to angle D, and angle B equal to angle E.

To prove that $\triangle ABC \cong \triangle DEF$.

Proof. Method, superposition.

Arguments	Reasons
Place $\triangle ABC$ on $\triangle DEF$, so that A falls on D and AB falls along DE.	
1. Then B will fall on E,	1. For AB is given = DE.
2. AC will fall along DF,	2. For $\angle A$ is given = $\angle D$.
3. and BC will fall along EF,	3. For $\angle B$ is given = $\angle E$.
4. Then, C must fall on F,	4. For two straight lines can intersect in only one point.
5. $\therefore \triangle ABC \cong \triangle DEF$.	5. Two figures that coincide throughout are congruent.

VII. Abbreviations for Theorems

These two theorems must be quoted as authority many times in proving triangles congruent; therefore, it is con-

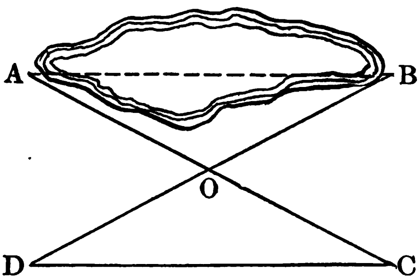
venient to abbreviate them in writing the reasons. The first theorem, about two sides and the included angle may be represented by $S - \angle - S$, in which S stands for a side and the usual symbol for angle standing between two S's shows the angle is included between the two sides. The second theorem about two angles and the included side, may be represented by $\angle - S - \angle$. Why should the S be placed between the two angle symbols?

VIII. Exercises

1. Draw a straight line AB. At A draw a line perpendicular to AB. At B draw a line perpendicular to AB but on the opposite side of AB. Through O, the midpoint of AB draw any line, cutting the perpendiculars in any points, C and D, respectively.

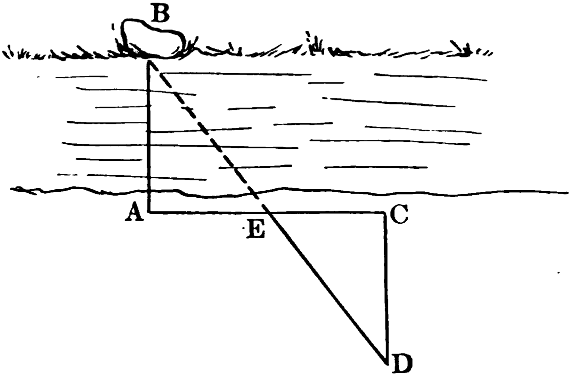
(a) Prove AC equals BD.
(b) Prove OC equals OD.
(c) Prove $\angle C = \angle D$.

2. Some boys wished to measure the length of a pond AB. With a tape they measured the line AC. From B, through O, the midpoint of AC, they ran the line BD, making OD equal to OB. Show that by measuring CD, they have the measure of the pond.



3. To measure the width of a river at A, boys sighted a bowlder across the river at B. One walked 100 ft. along the bank to E in a line at right angles to AB, and the other continued another 100 ft. to C. At C, he made a right angled turn away from the river and walked until he reached a

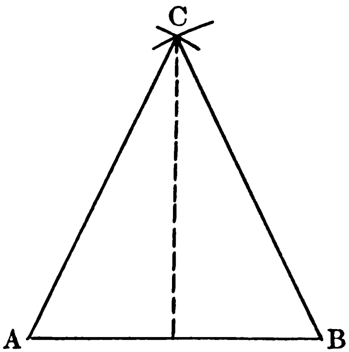
point D, which was in the same line with the first boy and the bowlder across the river. Which line may be measured



to give the width of the river? Prove that your reply is correct.

IX. Construction Lines in Proofs

Sometimes it is necessary to draw extra lines in a given figure in order to make two triangles that may be proved congruent. The isosceles triangle ABC may be divided into two triangles by drawing a line through C to bisect angle C. When a line is constructed to do one thing, it cannot be assumed to do anything else. If it is drawn to bisect angle C, it cannot be assumed to bisect AB nor to be perpendicular to AB. Such construction lines are usually drawn as dotted lines.

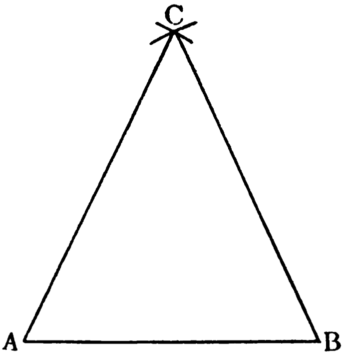


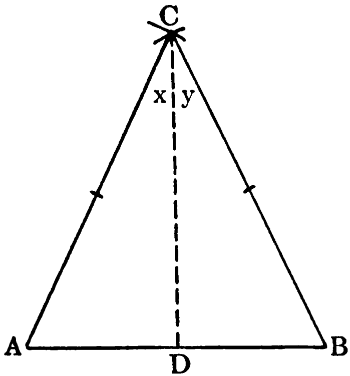
X. Demonstration

Proposition III

Theorem: ISOSCELES TRIANGLE

*In an isosceles triangle, the angles opposite the equal sides are equal.**





Given the isosceles triangle ABC, with AC equal to BC.
To prove $\angle A = \angle B$.
Proof. Method, $\cong$ ⚠, $S - \angle - S$.

Arguments	Reasons
Construction: Draw CD to bisect $\angle C$, meeting AB at D, to make two ⚠, ACD and BCD.	
In ⚠ACD and BCD,	
1. $AC = BC$,	1. Given as equal sides of the isos. $\triangle$.
2. $CD = CD$,	2. The common side.
3. and $\angle x = \angle y$.	3. Made = in const.
4. Then $\triangle ACD \cong \triangle BCD$.	4. $S - \angle - S$.
5. $\therefore \angle A = \angle B$.	5. Homologous parts of $\cong$ figures.

* This theorem is sometimes stated as: *The base angles of an isosceles triangle are equal.* It is abbreviated as, Base $\angle$s, isos. $\triangle$.

XI. Exercises

1. Show that the bisector of the vertex angle of an isosceles triangle is perpendicular to the base.

2. Show that the bisector of the vertex angle of an isosceles triangle bisects the base.

3. Draw a square $ABCD$ with the diagonal AC.

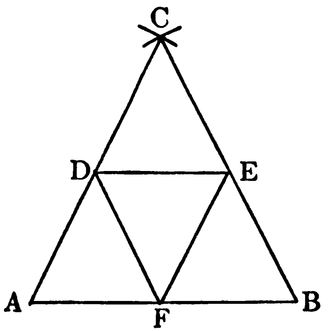
(a) Prove $\triangle ACD \cong \triangle ACB$.

(b) Prove $\angle BAC = \angle BCA$.

(c) Prove $\angle DAC = \angle DCA$.

(d) Show that AC bisects the angles of the square, A and C.

4. Draw an isosceles triangle, ABC. Join the midpoints of the three sides in succession, as DEF. Prove triangle DEF is isosceles.



5. (a) Draw a line AB. At A make any acute angle. At B, on the same side of AB, make an angle equal to angle A. Prolong the sides of the two angles until they intersect at C. Measure the sides AC and BC. What kind of triangle is ABC?

(b) If two angles of a triangle are equal, are the sides opposite them equal?

(c) In Proposition III, the *hypothesis* is, *two* **sides** *of a triangle are equal* and the *conclusion* is *the* **angles** *opposite these sides are equal.*

In 3 (b), the *hypothesis* is, *two* **angles** *of a triangle are equal* and the *conclusion* is *the* **sides** *opposite these angles are equal.*

(d) When a new theorem is made from another by interchanging the hypothesis and the conclusion, the one theorem is called the *converse* of the other. *Converse* comes from a Latin word which means *turned.*

(e) Because a theorem is true, it does not necessarily follow that its converse is true. The statement that *all horses are animals* is true, but its converse, *all animals are horses,* is not true. The converses of some theorems are true, as is the one about the angles of an isosceles triangle, but each must be proved.

(f) In a complete course in geometry this converse theorem is proved. In this abridged course, we shall accept our construction and experimentation as sufficient evidence of its truth.

6. (a) Draw a rectangle $ABCD$ with its diagonals intersecting at O.

(b) Prove that AOB is an isosceles triangle.

(c) Prove that three other triangles are isosceles.

(d) Prove that $\triangle AOB \cong \triangle DOC$.

(e) Prove that the diagonals of a rectangle bisect each other.

7. (a) Draw a line AB 6 cm. long.

(b) At each end of AB, construct an angle of 45°. What kind of triangle is formed?

(c) Draw the altitude of the triangle and compute its height.

(d) How long is each side of the triangle?

8. (a) Draw an angle of 60° and bisect it.

(b) Through any point in the bisector draw a line perpendicular to it and long enough to cut the sides of the angle.

(c) What kind of triangle is formed?

XII. Experiments

1. Draw a triangle ABC.

2. Draw another triangle DEF, making DE equal to AB, DF equal to AC, and EF equal to BC.

3. Cut out triangle ABC and place it on triangle DEF, so that the equal bases coincide, and with F and C on the same side of the bases.

4. Since you do not know that angle D equals angle A, may you claim that AC falls along DF?

Why may you not claim that BC falls along EF?

5. Why is it impossible to prove these triangles congruent by the usual method of superposition?

6. Try placing these two triangles with their bases together, but with their vertices, C and F, on opposite sides of the base.

7. The figure now has four sides. Find two equal sides. What line can you draw that will make an isosceles triangle with these two equal sides?

8. Draw the line and find another isosceles triangle in the figure.

9. In each isosceles triangle, which angles are equal?

10. Which of these angles added together make angle C? angle F?

11. How can you prove angle C equals angle F?

12. Which parts of triangle ABC are now known to equal the corresponding parts of triangle DEF?

13. Which of the first two theorems on congruent triangles can be used to prove triangle ABC congruent to triangle DEF?

14. What is your conclusion about the congruence of two triangles that have three sides of one equal respectively to three sides of the other?

15. Write out a complete proof. Then compare it with the one on the next page.

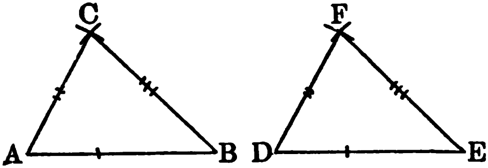
XIII. Demonstration

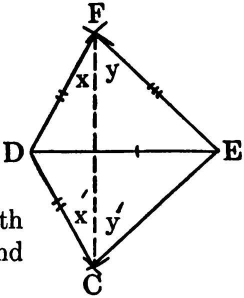
Proposition IV

CONGRUENCE OF TRIANGLES

Theorem: Three sides.

Two triangles are congruent if three sides of one are equal respectively to three sides of the other.





Given the triangles ABC and DEF, with AB equal to DE, AC equal to DF, and BC equal to EF.

To prove $\triangle ABC \cong \triangle DEF$.

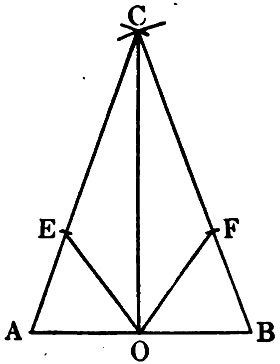
Proof. Method, isos. $\triangle$ and $S - \angle - S$.

Arguments	Reasons
Construction: Let AB and DE be the longest sides of the $\triangle$s. Place $\triangle ABC$ next to $\triangle DEF$, with A on D, and with AB falling along DE, and with C opposite F.	
1. Then B will fall on E.	1. For AB is given $= D$.
Draw the line CF.	
2. Then FDC is an isos. $\triangle$,	2. Why?
3. and $\angle x = \angle x'$.	3. Why?
4. $\triangle FEC$ is an isos. $\triangle$,	4. Why?
5. and $\angle y = \angle y'$.	5. Why?
6. Then $\angle x + \angle y = \angle x' + \angle y'$,	6. Add. Ax.
7. Or $\angle F = \angle C$.	7. The whole = sum of its parts.
8. $\therefore \triangle ABC$ (or $\triangle DEC$) $\cong \triangle DEF$.	8. $S - \angle - S$.

Note. — The abbreviation for the theorem about three sides is $S - S - S$.

XIV. Exercises

1. Bisect an angle and prove your construction correct.
2. Prove that a diagonal of a square bisects two angles of the square.



3. (a) Prove that a line drawn from the vertex of an isosceles triangle to the mid-point of the base divides the triangle into two congruent triangles.
   (b) Prove that this line is perpendicular to the base.
4. ABC is an isosceles triangle. On the two legs take $AE = BF$. Draw lines from O, the mid-point of the base, to E and F.
   (a) Prove that $OE = OF$.
   (b) Draw CO and prove triangle COE congruent to triangle COF.
5. (a) Draw a square and connect the midpoints of the sides in succession.
   (b) Prove that the four triangles at the corners are congruent.
6. (a) Draw a rectangle and connect the midpoints of the sides in succession.
   (b) Prove that the four triangles at the corners are congruent.
   (c) What is the inner figure?
7. (a) Draw a rhombus with one diagonal.
   (b) Prove that the diagonal divides the rhombus into two congruent triangles.
8. (a) On opposite sides of a line AB, draw two unequal isosceles triangles, ABC and ABD.
   (b) Prove that $\angle CAD = \angle CBD$.

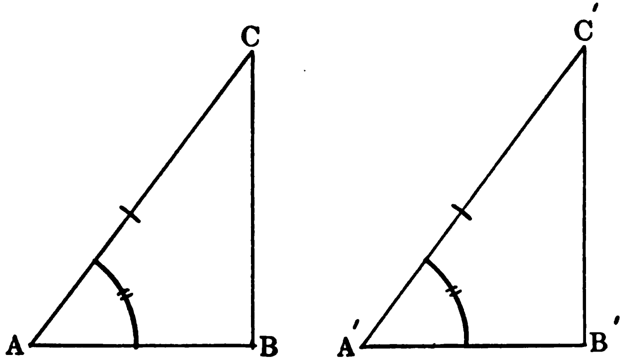
XV. Demonstration

PROPOSITION V

CONGRUENCE OF RIGHT TRIANGLES

Theorem: HYPOTENUSE AND ACUTE ANGLE.

Two right triangles are congruent if the hypotenuse and an acute angle of one are equal respectively to the hypotenuse and an acute angle of the other.



Given the right triangles ABC and $A'B'C'$, with the hypotenuse AC equal to the hypotenuse $A'C'$ and with angle A equal to angle A'.

To prove $\triangle ABC \cong \triangle A'B'C'$.

Proof. Method, superposition.

Arguments	Reasons
Place $\triangle ABC$ on $\triangle A'B'C'$ with A on A' and AC falling along $A'C'$.	
1. Then C will fall on C',	1. For AC is given $= A'C'$.
2. and AB will fall along $A'B'$.	2. For $\angle A$ is given $= \angle A'$.
3. Now B and B' are rt. $\angle$s.	3. For $\triangle$s ABC and $A'B'C'$ are given as rt. $\triangle$s.
4. Then CB must coincide with $C'B'$.	4. Only one $\perp$ can be drawn from C' to $A'B'$.
5. $\therefore \triangle ABC \cong \triangle A'B'C'$.	5. For they coincide throughout.

NOTE. — The abbreviation for this theorem is $H - \angle$.

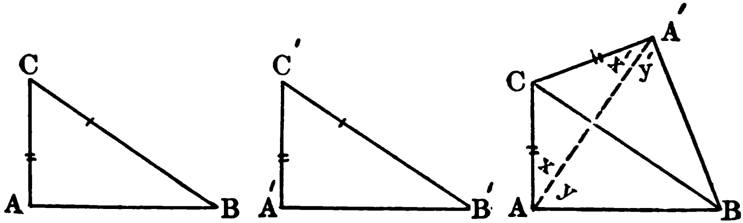
XVI. Demonstration

Proposition VI

CONGRUENCE OF RIGHT TRIANGLES

Theorem: HYPOTENUSE AND LEG.

Two right triangles are congruent if the hypotenuse and leg of one are equal respectively to the hypotenuse and leg of the other.



Given the right triangles ABC and $A'B'C'$ with the hypotenuse BC equal to the hypotenuse $B'C'$ and the leg AC equal to the leg $A'C'$.

To prove that $\triangle ABC \cong \triangle A'B'C'$.

Proof. Method, adjacent placement, isos. $\triangle$, and $S-S-S$. (Why cannot superposition be used?)

Arguments	Reasons
Place $\triangle A'B'C'$ next to $\triangle ABC$, so that C' falls on C, $C'B'$ falls along B, and A' is opposite A.	
1. Then B' will fall on B. Draw AA'.	1. For $C'B'$ is given $= CB$.
2. $\triangle ACA'$ is isos.	2. For $CA = CA'$, given as equal legs of rt $\triangle$s.
3. Then, $\angle x = \angle x'$.	3. Base $\angle$s of isos. $\triangle ACA'$.
4. But $\angle A = \angle A'$.	4. Rt. $\angle$s of given rt. $\triangle$s.
By subtracting (3) from (4),	
5. $\angle A - \angle x = \angle A' - \angle x'$	5. Subtraction Ax.
6. or $\angle y = \angle y'$	6. Substitution of equals for equals.
7. $\therefore AB = A'B$.	7. Sides opp. $=$ $\angle$s of a $\triangle$ are $=$.
8. $\triangle ABC \cong \triangle A'B'C'$ (or $A'BC$).	8. $S-S-S$.

NOTE. — The abbreviation for this theorem is $H-L$.

XVII. Exercises

1. (a) Construct a bisector of a given line AB.

(b) Prove your construction correct.

2. Draw an isosceles triangle ABC, on AB as a base. From A, draw AX perpendicular to BC. From B, draw BY perpendicular to AC.

(a) Prove $\triangle ABX = \triangle ABY$. (c) Prove $\angle BAX = \angle ABY$.
(b) Prove $AX = BY$. (d) Prove $\triangle ACX = \triangle BCY$.

3. Draw an isosceles triangle ABC on AB as a base. From O, the mid-point of AB, draw lines perpendicular to the two legs, meeting AC at D and BC at E.

(a) Prove $OD = OE$. (c) Prove $\angle AOD = \angle BOE$.
(b) Prove $AD = BE$. (d) Prove $\angle DOE = 2$ times $\angle A$.
(e) Prove $\angle C = \angle AOD + \angle BOE$.

H. A STUDY OF PARALLEL LINES

The next group of theorems is concerning parallel lines.

In Book Two, by experimentation, you established, to your own satisfaction, the truth of several theorems concerning parallel lines. In a complete course in geometry, these theorems would have to be proved. In this course, however, we may use them as assumptions.

I. Assumptions Concerning Parallels

Make a list of assumptions concerning parallels by answering the following questions.

1. Through a given point, how many lines can be drawn parallel to a given line?

This assumption, though readily accepted, cannot be proved. All the others may be.

2. If two lines are both perpendicular to the same straight line, what is their relation to each other?

3. If a line is perpendicular to one of two parallel lines what is its relation to the other?

II. Angles Made by a Transversal

1. Draw two lines (not parallel) and draw a third line cutting the other two. What is the name of the third line?

2. In the figure of two lines cut by a transversal, name all of the exterior angles; the interior angles; the corresponding angles; the alternate interior angles; the alternate exterior angles.

III. Experiments

1. Draw two parallel lines cut by a transversal. Apparently, how do the alternate interior angles compare? How would you proceed to prove them equal?

2. The first thing to do is to see if the two angles are in triangles that may be proved congruent.

3. If not, can a line be drawn to make two triangles?

4. Any line drawn through the transversal to the parallels would make two triangles. Through which point of the transversal should the line be drawn if the sides of the two triangles are to be equal? Which two angles are equal?

5. How many parts of the two triangles must be equal respectively to prove the triangles are congruent? Since only two such parts can be found, it is impossible to prove the triangles congruent unless we can use one of the right triangle theorems, which require only two parts. Try drawing the line bisecting the transversal so as to make right triangles.

6. Can this line be drawn perpendicular to both parallels? Can it be drawn perpendicular to one and proved perpendicular to the other? If so, how may the two right triangles be proved congruent?

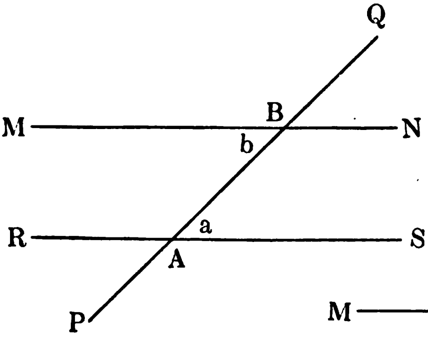
7. Write out a complete proof. Then compare with the one given on the next page.

IV. Demonstration

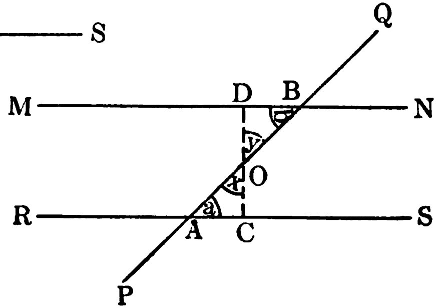
Proposition VII

PARALLEL LINES

Theorem: ALTERNATE INTERIOR ANGLES.



If two parallel lines are cut by a transversal, the alternate interior angles are equal.



Given two parallel lines RS and MN cut by the transversal PQ, forming the alternate interior angles, a and b.

To prove $\angle a = \angle b$.

Proof. Method, rt. ▲, $H - \angle$.

Arguments	Reasons
Through O, the midpoint of AB, draw $OC \perp RS$. Prolong OC to meet MN at D.	
1. Then $CD \perp MN$.	1. A line $\perp$ to one of two $\parallel$s, is $\perp$ to the other also.
2. AOC and BOD are rt. ▲, In these ▲,	2. For ∠s C and D are rt. ∠s.
3. $AO = OB$.	3. By const., O is midpoint.
4. $\angle x = \angle y$.	4. Vertical ∠s.
5. Then, $\triangle AOC \cong \triangle BOD$.	5. Rt. ▲, $H - \angle$.
6. $\therefore \angle a = \angle b$.	6. Homologous ∠s.

Note. — The abbreviation for this theorem is *alt. int.* ∠s *of* $\parallel$s.

V. Exercises

1. Prove that the opposite sides of a parallelogram are equal.

Suggestions:

(a) Draw one diagonal.

(b) By the definition of a parallelogram, what relation between the opposite sides is given? What parts of the triangles, therefore, are equal?

2. Draw a parallelogram with two diagonals.

(a) Prove two pairs of triangles are equal.

(b) Prove that the diagonals bisect each other.

3. (a) Draw a parallelogram with its two diameters or midjoins.

(b) Join the ends of the diameters in succession.

(c) Prove that the figure formed is a rhombus.

A Complete Course in Geometry

The few theorems given in this chapter illustrate the kind of reasoning used in the demonstration of geometrical propositions. The complete course in geometry includes other methods of proof and many other theorems concerning all kinds of geometric figures.

REVIEW QUESTIONS

On Chapter One

1. Define the following terms: positive, negative, horizontal scale, vertical scale.

2. (a) How are two positive numbers added? two negative numbers? a positive and a negative number?
   (b) How is the sign of the sum determined?

3. How is the sign of the product of two numbers determined?

4. What are factors of a number? of an algebraic expression?

5. How many factors are in the area of a surface? Name them.

6. How many factors are in the volume of a solid? Name them.

7. (a) When is the product of two numbers less than one of the numbers?
   (b) When is the product of two numbers less than both of them?

On Chapter Two

1. (a) Define the following terms: first degree expression, second degree expression, quadratic expression, cubic expression.
   (b) Give another name for cubic expression.

2. (a) What is meant by general numbers?
   (b) Show some uses and advantages of general numbers.

3. (a) Define axiom.
   (b) Name and give six axioms.

4. What advantages do you have in studying algebra to-day over those of a school boy of the fifteenth or sixteenth century?

On Chapter Three

1. Define transposition, simple equation, root of an equation, variable.

2. What is the graph of a simple equation?

3. What is a linear equation? Why is it so called?

4. What is meant by an equation with two variables?

5. What is a quadratic equation?

6. How many roots does a quadratic equation have? Show why.

7. Distinguish between a complete and an incomplete quadratic equation.

8. Make a problem that gives a complete quadratic equation; an incomplete quadratic.

On Chapter Four

1. What is the difference between subtraction in arithmetic and in algebra?

2. Compare the laws of signs in multiplication with those in division.

3. Of what importance is the order of terms in division?

On Chapter Five

1. What is the effect of a parenthesis?

2. How may parentheses be removed?

On Chapter Six

1. What kind of fractions may be added?

2. How may fractions be changed to a common denominator?

3. How may fractions be reduced to lowest terms? Why?

4. What changes in the signs of a fraction are permissible? Why?

On Chapter Seven

1. In multiplication and division of fractions, what precaution should be taken in cancellation?

2. (a) What is a complex fraction?
(b) How may it be simplified?

On Chapter Eight

1. What is a system of equations?

2. What determines the number of equations in a system?

3. What is the significance of the word *simultaneous* in reference to a system of equations?

4. How does the graph of a system of linear equations give the roots?

5. What are coördinates of a point?

6. What contributions did Descartes make to mathematics?

7. (a) What are two methods for solving a system of linear equations algebraically?
(b) Under what conditions is each preferable?

8. What is the standard form of a linear equation of two variables?

On Chapter Nine

1. What is the standard form of a quadratic equation of one variable?

2. (a) Name four ways of solving a quadratic equation.
(b) What are the advantages of each?

3. How does the graph of a quadratic equation compare with that of a linear?

4. What is a parabola?

5. How does the graph of a quadratic equation show the number of its roots?

6. What is a quadratic surd?

On Chapter Ten

1. What is a system of simultaneous quadratic equations?
2. Name and draw different curves that are the graphs of quadratic equations.

On Chapter Eleven

1. Define similar figures, homologous angles, tangent, infinity, sine, cosine, complement, reciprocal, cotangent, function, trigonometry.
2. What are some of the advantages of trigonometry?

On Chapter Twelve

1. What are exponents?
2. What is the law of exponents in multiplication? in division?
3. What is the effect of zero as an exponent?
4. What are logarithms?
5. What is the characteristic of a logarithm?
6. How is the characteristic determined?
7. What is the mantissa?
8. Define antilogarithm.
9. Who invented logarithms?
10. What are their advantages and practical uses?

On Chapter Fourteen

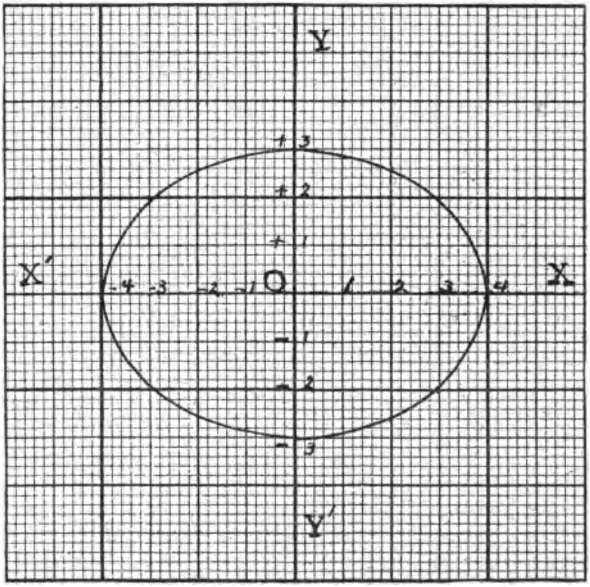
1. What is the meaning of the word *geometry*?
2. Name five geometers among the early Greeks and give at least one contribution of each.
3. Define major premise, minor premise, conclusion, congruent figures, theorem, superposition, hypothesis.
4. State three general theorems about congruent triangles.
5. State two theorems about congruent right triangles.
6. Debate the following: Euclid's contribution to mathematics is greater than that of Pythagoras.

APPENDIX

A. OTHER CURVES OF QUADRATIC EQUATIONS IN TWO UNKNOWNS

I. The Ellipse

1. (a) Another kind of curve is found in the next equation. It is an *ellipse.*



GRAPH OF $9x^2 + 16y^2 = 144$

(b) How does it differ from a circle?

(c) Draw the graph of

$$9x^2 + 16y^2 = 144. \qquad (1)$$

Transposing, $$16y^2 = 144 - 9x^2. \qquad (2)$$

From (2), $$y^2 = \frac{144 - 9x^2}{16}. \quad (3)$$

$$\therefore y = \pm\sqrt{\frac{9(16 - x^2)}{16}} \quad (4)$$

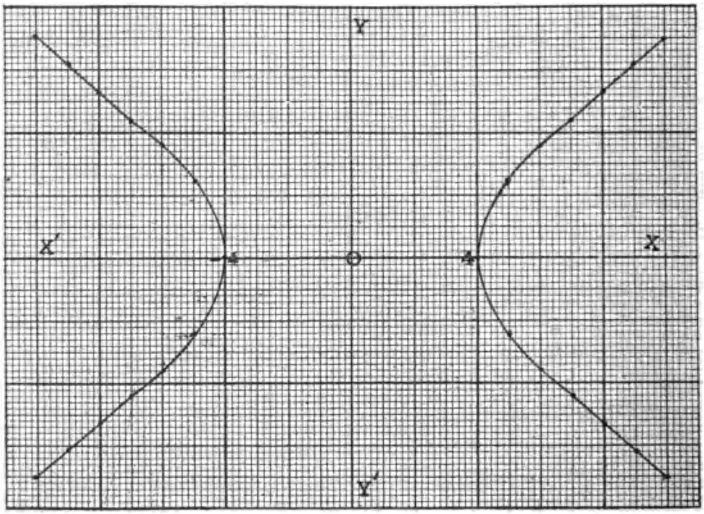
$$= \pm \tfrac{3}{4}\sqrt{16 - x^2} \quad (5)$$

$9x^2 + 16y^2 = 144$

If x is	then y is
0	$\pm$ 3
1	$\pm$ 2.9
2	$\pm$ 2.6
3	$\pm$ 1.9
4	0
$-$ 1	$\pm$ 2.9
$-$ 3	$\pm$ 1.9
$-$ 4	0

II. The Hyperbola Cutting the X-Axis

1. (a) If the sign between the two quadratic terms is minus, the curve is entirely different from the closed ellipse.



GRAPH OF $9x^2 - 16y^2 = 144$

(b) It consists of two parts, one on each side of the y-axis. It cannot be a closed curve, for the sides of each arm grow farther apart as the x-distance increases.

(c) This curve is called an *hyperbola.*

(d) Draw the graph of $9x^2 - 16y^2 = 144$.

$9x^2 - 16y^2 = 144$

If x is	then y is
4	0
5	$\pm 2\frac{1}{4}$
6	± 3.4
7	± 4.3
8	± 5.2
9	± 6.1
10	± 6.9
-4	0
-5	$\pm 2\frac{1}{4}$
-6	± 3.4
-7	± 4.3
-8	± 5.2
-9	± 6.1
-10	± 6.9

III. The Hyperbola in Opposite Quadrants

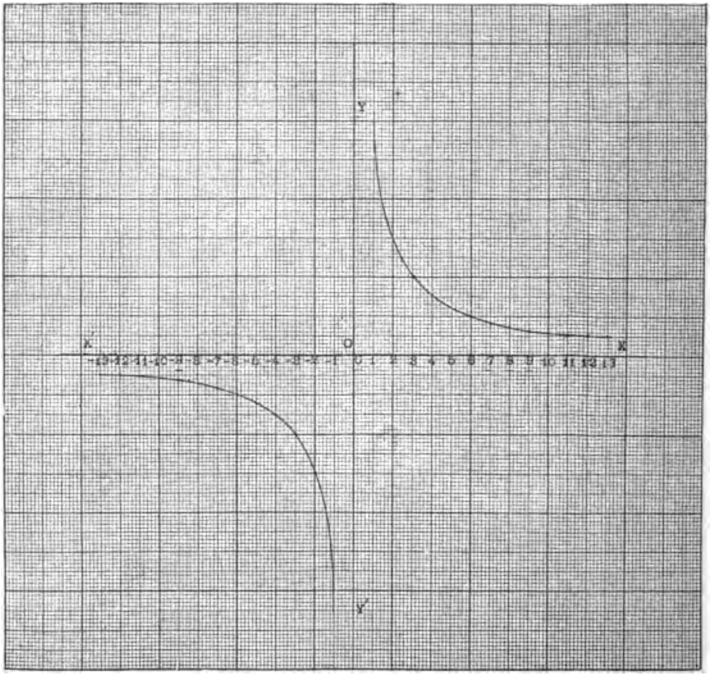
1. (a) If the product of two numbers is 12, the equation of this statement is $xy = 12$. This equation is quadratic because the term xy has two unknown factors.

(b) The graph of $xy = 12$ gives an hyperbola also, but instead of cutting the axis, the two arms are in opposite quadrants.

(c) Graph $xy = 12$.

$$xy = 12$$

If x is	then y is	If x is	then y is
1	12	9	1.3
2	6	10	1.2
3	4	11	1.1
4	3	12	1.
5	2.4	15	.8
6	2	-1	-12
7	1.7	-2	-6
8	1.5	-3	-2.4
		etc.	



Graph of $xy = 12$

B. TABLE OF POWERS AND ROOTS OF NUMBERS

No.	Squares	Cubes	Square Roots	Cube Roots	No.	Squares	Cubes	Square Roots	Cube Roots
1	1	1	1.000	1.000	51	2,601	132,651	7.141	3.708
2	4	8	1.414	1.259	52	2,704	140,608	7.211	3.732
3	9	27	1.732	1.442	53	2,809	148,877	7.280	3.756
4	16	64	2.000	1.587	54	2,916	157,464	7.348	3.779
5	25	125	2.236	1.709	55	3,025	166,375	7.416	3.802
6	36	216	2.449	1.817	56	3,136	175,616	7.483	3.825
7	49	343	2.645	1.912	57	3,249	185,193	7.549	3.848
8	64	512	2.828	2.000	58	3,364	195,112	7.615	3.870
9	81	729	3.000	2.080	59	3,481	205,379	7.681	3.892
10	100	1,000	3.162	2.154	60	3,600	216,000	7.745	3.914
11	121	1,331	3.316	2.223	61	3,721	226,981	7.810	3.936
12	144	1,728	3.464	2.289	62	3,844	238,328	7.874	3.957
13	169	2,197	3.605	2.351	63	3,969	250,047	7.937	3.979
14	196	2,744	3.741	2.410	64	4,096	262,144	8.000	4.000
15	225	3,375	3.872	2.466	65	4,225	274,625	8.062	4.020
16	256	4,096	4.000	2.519	66	4,356	287,496	8.124	4.041
17	289	4,913	4.123	2.571	67	4,489	300,763	8.185	4.061
18	324	5,832	4.242	2.620	68	4,624	314,432	8.246	4.081
19	361	6,859	4.358	2.668	69	4,761	328,509	8.306	4.101
20	400	8,000	4.472	2.714	70	4,900	343,000	8.366	4.121
21	441	9,261	4.582	2.758	71	5,041	357,911	8.426	4.140
22	484	10,648	4.690	2.802	72	5,184	373,248	8.485	4.160
23	529	12,167	4.795	2.843	73	5,329	389,017	8.544	4.179
24	576	13,824	4.898	2.884	74	5,476	405,224	8.602	4.198
25	625	15,625	5.000	2.924	75	5,625	421,875	8.660	4.217
26	676	17,576	5.099	2.962	76	5,776	438,976	8.717	4.235
27	729	19,683	5.196	3.000	77	5,929	456,533	8.774	4.254
28	784	21,952	5.291	3.036	78	6,084	474,552	8.831	4.272
29	841	24,389	5.385	3.072	79	6,241	493,039	8.888	4.290
30	900	27,000	5.477	3.107	80	6,400	512,000	8.944	4.308
31	961	29,791	5.567	3.141	81	6,561	531,441	9.000	4.326
32	1,024	32,768	5.656	3.174	82	6,724	551,368	9.055	4.344
33	1,089	35,937	5.744	3.207	83	6,889	571,787	9.110	4.362
34	1,156	30,304	5.830	3.239	84	7,056	592,704	9.165	4.379
35	1,225	42,875	5.916	3.271	85	7,225	614,125	9.219	4.396
36	1,296	46,656	6.000	3.301	86	7,396	636,056	9.273	4.414
37	1,369	50,653	6.082	3.332	87	7,569	658,503	9.327	4.431
38	1,444	54,872	6.164	3.361	88	7,744	681,472	9.380	4.447
39	1,521	59,319	6.244	3.391	89	7,921	704,969	9.433	4.464
40	1,600	64,000	6.324	3.419	90	8,100	729,000	9.486	4.481
41	1,681	68,921	6.403	3.448	91	8,281	753,571	9.539	4.497
42	1,764	74,088	6.480	3.476	92	8,464	778,688	9.591	4.514
43	1,849	79,507	6.557	3.503	93	8,649	804,357	9.643	4.530
44	1,936	85,184	6.633	3.530	94	8,836	830,584	9.695	4.546
45	2,025	91,125	6.708	3.556	95	9,025	857,375	9.746	4.562
46	2,116	97,336	6.782	3.583	96	9,216	884,736	9.797	4.578
47	2,209	103,823	6.855	3.608	97	9,409	912,673	9.848	4.594
48	2,304	110,592	6.928	3.634	98	9,604	941,192	9.899	4.610
49	2,401	117,649	7.000	3.659	99	9,801	970,299	9.949	4.626
50	2,500	125,000	7.071	3.684	100	10,000	1,000,000	10.000	4.641

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